

A sum of three squares in which one is the product of the other two

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Abstract

The identity $x^2 + (x+1)^2 + (x(x+1))^2 = (x^2 + x + 1)^2$ is the motivation for study of the equation $d^2 + e^2 + n^2 = x^2$, where d and e are conjugate divisors of n , that is, $n = de$ and $d < e$. Let N_n be the set of solutions to the given equation. Solutions with $|N_n| > 1$ fall empirically into two families: a triangular family $\mathcal{T}$, and a Fibonacci family $\mathcal{F}$, with virtually no overlap. It is conjectured that the two families comprise all n with $|N_n| > 1$.

Disclaimer: The CAS *Maple* was used extensively.

Duodecimal declaration: The author uses duodecimal, base 12, in the body of this document. Document data is in base 10, mostly because not *all* document data could be converted robustly to base 12.

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1 Introduction

The well-known identity

$$(1) \quad x^2 + (x+1)^2 + (x(x+1))^2 = (x^2 + x + 1)^2.$$

is the motivation for the study of the Diophantine equation

$$(2) \quad d^2 + e^2 + n^2 = x^2,$$

where $n = de$, $d < e$. By Lagrange's four-square theorem, n cannot be of the form $4^k(8m+7)$. Let $n \geq 2$, let $D_n = \{d|n : d^2 < n\}$, and let D_n^2 be the set of ordered pairs (d, e) such that $d \in D_n$ and $e = n/d$. Define

$$(3) \quad N(n) = \{[d, e, n, x] : d^2 + e^2 + n^2 = x^2, (d, e) \in D_n^2\}.$$

The set N_n is ordered by decreasing d . The set of all n such that $|N(n)| > 1$ is the topic of this paper. Table 2 shows all solutions to (2) with $|N_n| > 1$ and $n \leq 10^4$. Empirically, it seems that, if $|N_n| > 1$, then $|N_n| = 2$, $n \neq 180$, and $|N_{180}| = 3$. Furthermore, N_n has two distinct families: a triangular family $\mathcal{T}$, and a Fibonacci family $\mathcal{F}$, with only a little overlap. The author has been unable to prove necessity of the two families.

1.1 Remark. Sequences occurring in this document are in Section 6, and are referenced with the prefix 'S.'

2 The triangular family

See Table 3 for the solutions of (2) consisting of those n for which $|N_n| > 1$, and with one solution having d and e consecutive. The following are verified by direct calculation.

► **The triangular family, $\mathcal{T}$.** 1. Let $T_k = k(k+1)/2$ be a triangular number. Then

$$(4a) \quad T_k^2 + (T_k + 1)^2 + (T_k(T_k + 1))^2 = (T_k^2 + T_k + 1)^2 = x_1^2,$$

$$(4b) \quad x_1 = (k^4 + 2k^3 + 3k^2 + 2k + 4)/4.$$

2. If k is odd, then

$$(5a) \quad ((k+1)/2)^2 + (k(T_k + 1))^2 + (T_k(T_k + 1))^2 = x_2^2,$$

$$(5b) \quad x_2 = (k^4 + 2k^3 + 5k^2 + 2k + 2)/4.$$

3. If k is even, then

$$(6a) \quad (k/2)^2 + ((k+1)(T_k + 1))^2 + (T_k(T_k + 1))^2 = x_3^2,$$

$$(6b) \quad x_3 = (k^4 + 2k^3 + 5k^2 + 6k + 4)/4.$$

4. Define $\mathcal{T}_n = N_n$, where $n = T_k(T_k + 1)$ for some triangular number T_k , and empty otherwise.

2.1 Remark. See Table 3. The following sequences are in the OEIS.

1. S.6.5 [4], $a_k = T_k$.
2. S.6.7, $a_k = T_k(T_k + 1)$.
3. S.6.9, $a(k) = kT_k$.
4. S.6.10, Nonnegative integers repeated.

2.2 Remark. The following sequences are in the OEIS.

1. S.6.12, $a(k) = (k+1)(T_k+1)$.
2. The sequence $a(k) = x_1(k)$ is not in the OEIS.
3. The sequence $a(k) = x_2(k)$ is not in the OEIS.
4. The sequence $a(k) = x_3(k)$ is not in the OEIS.

The triangular family $\mathcal{T}$ can also be written as a hexagonal family $\mathcal{H}$. We have

● **Hexagonal family $\mathcal{H}$.** 1. Let $n = H1_k(H1_k+1)$, where $H1_k = T_{2k-1} = k(2k-1)$, the odd triangular numbers, also the sequence of first hexagonal numbers. Then

$$(7a) \quad H1_k^2 + (H1_k+1)^2 + (H1_k(H1_k+1))^2 = (H1_k^2 + H1_k+1) = x_4^2,$$

$$(7b) \quad x_4 = 4k^4 - 4k^3 + 3k^2 - k + 1,$$

$$(7c) \quad k^2 + ((2k-1)(H1_k+1))^2 + (H1_k(H1_k+1))^2 = x_5^2,$$

$$(7d) \quad x_5 = 4k^4 - 4k^3 + 5k^2 - 3k + 1;$$

2. Let $n = H2_k(H2_k+1)$, where $H2_k = T_{2k} = k(2k+1)$, the even triangular numbers, also the sequence of second hexagonal numbers. Then

$$(8a) \quad H2_k^2 + (H2_k+1)^2 + (H2_k(H2_k+1))^2 = (H2_k^2 + H2_k+1) = x_6^2,$$

$$(8b) \quad x_6 = 4k^4 + 4k^3 + 3k^2 + k + 1,$$

$$(8c) \quad k^2 + ((2k+1)(H2_k+1))^2 + (H2_k(H2_k+1))^2 = x_7^2,$$

$$(8d) \quad x_7 = 4k^4 + 4k^3 + 5k^2 + 3k + 1.$$

2.3 Remark. The following sequences are in the OEIS.

1. S.6.6, $a_k = H1_k = k(2k-1)$, the first hexagonal numbers;
2. S.6.13, $a_k = H2_k = k(2k+1)$, the second hexagonal numbers.

2.4 Remark. The following sequences are not in the OEIS.

- | | |
|-----------------------------|----------------------|
| 1. $a_k = H1_k(H1_k+1)$; | 5. $a(k) = x_4(k)$; |
| 2. $a_k = H2_k(H2_k+1)$; | 6. $a(k) = x_5(k)$; |
| 3. $a_k = (2n-1)(H1_k+1)$; | 7. $a(k) = x_6(k)$; |
| 4. $a_k = (2k+1)(H2_k+1)$; | 8. $a(k) = x_7(k)$. |

3 The Fibonacci family

If one searches for multiple solutions to (2) for fixed n , where no d and e are consecutive, then one finds the pairs of solutions displayed in Table 4, which contains the following sequences, obtained from the OEIS.

3.1 Remark. The following sequences are in the OEIS.

1. S.6.3, $a(k) = F_k$, the Fibonacci sequence [2, 3];
2. S.6.8, $a(k) = F_{2k}$;
3. S.6.14, $a(k) = 2F_{2k-1}F_{2k}$.

Thus, we have the following:

◆ **The Fibonacci family, $\mathcal{F}$.** Let F_k denote a general term of the Fibonacci sequence. Then we have the following identities.

$$(9a) \quad F_{2k}^2 + (2F_{2k+1}F_{2k+2})^2 + (2F_{2k}F_{2k+1}F_{2k+2})^2 = (F_{2k+3} + 2F_{2k}F_{2k+1}F_{2k+2})^2,$$

$$(9b) \quad F_{2k+2}^2 + (2F_{2k}F_{2k+1})^2 + (2F_{2k}F_{2k+1}F_{2k+2})^2 = (F_{2k-1} + 2F_{2k}F_{2k+1}F_{2k+2})^2.$$

Define $\mathcal{F}_n = N_n$, where $n = 2F_{2k}F_{2k+1}F_{2k+2}$, and empty otherwise.

3.2 Remark. The following sequences are not in the OEIS.

1. $a(k) = 2F_{2k}F_{2k+1}F_{2k+2}$;
2. $a(k) = F_{2k-1} + 2F_{2k}F_{2k+1}F_{2k+2}$;
3. $a(k) = F_{2k+3} + 2F_{2k}F_{2k+1}F_{2k+2}$.

It is easier to prove (9b) before (9a).

Proof of (9b). Let $n = 2k$, and let

$$\begin{aligned} a &= F_n, & b &= F_{n+1}, & c &= F_{n+2}, & d &= F_{n-1}, \\ c &= a + b, & d &= b - a. \end{aligned}$$

The identity (9b) becomes

$$E \triangleq c^2 + (2ab)^2 + (2abc)^2 - (d + 2abc)^2.$$

Simplifying, we obtain

$$\begin{aligned} E &= c^2 + 4a^2b^2 + 4a^2b^2c^2 - d^2 - 4abcd - 4a^2b^2c^2, \\ E &= c^2 + 4a^2b^2 - d^2 - 4abcd, \end{aligned}$$

Substituting again and simplifying, we obtain

$$\begin{aligned} E &= (a + b)^2 + 4a^2b^2 - (b - a)^2 - 4ab(a + b)(b - a), \\ &= a^2 + 2ab + b^2 + 4a^2b^2 - b^2 + 2ab - a^2 - 4ab(b^2 - a^2), \\ &= 4ab + 4a^2b^2 - 4ab^3 + 4a^3b, \\ &= 4a^3b + 4a^2b^2 - 4ab^3 + 4ab, \\ (10) \quad E &= -4ab(b^2 - ab - a^2 - 1). \end{aligned}$$

It remains to verify that $E = 0$. Recall Cassini's identity: $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$. Replace n by $n+1$ and use the fact that n is even to obtain $F_{n+1}^2 - F_nF_{n+2} = 1$, or equivalently, $b^2 - a(a+b) = 1$. Consequently, we have

$$(11) \quad b^2 - ab - a^2 = 1.$$

Using (11) in (10) we obtain $E = 0$, and so (9b) is proven. $\square$

Proof of (9a). The proof is similar to the proof of (9b). Let $n = 2k$ and

$$(12) \quad a = F_n, \quad b = F_{n+1}, \quad c = F_{n+2}, \quad d = F_{n+3},$$

$$(13) \quad c = a + b, \quad d = a + 2b.$$

Setting (9a) to zero and substituting, we obtain

$$\begin{aligned} E &\triangleq a^2 + (2bc)^2 + (2abc)^2 - (d + 2abc)^2, \\ &= a^2 + 4b^2c^2 + 4a^2b^2c^2 - d^2 - 4abcd - 4a^2b^2c^2, \\ &= a^2 + 4b^2c^2 - d^2 - 4abcd, \\ &= a^2 + 4b^2(a+b)^2 - (a+2b)^2 - 4ab(a+b)(a+2b), \\ &= a^2 + 4b^2(a^2 + 2ab + b^2) - (a^2 + 4ab + 4b^2) - 4ab(a^2 + 3ab + 2b^2), \\ &= a^2 + 4a^2b^2 + 8ab^3 + 4b^4 - a^2 - 4ab - 4b^2 - 4a^3b - 12a^2b^2 - 8ab^3, \\ &= 4b^4 - 4ab - 4b^2 - 4a^3b - 8a^2b^2, \\ &= 4b^4 - 4a^3b - 8a^2b^2 - 4b^2 - 4ab, \\ (14) \quad E &= 4b(b^3 - 2a^2b - a^3 - a - b). \end{aligned}$$

It remains to verify that $E = 0$. Multiply (11) by a and then by b to obtain the pair of equations

$$(15) \quad ab^2 - a^2b - a^3 = a,$$

$$(16) \quad b^3 - ab^2 - a^2b = b.$$

Adding (15) and (16), we obtain

$$b^3 - 2a^2b - a^3 = a + b,$$

whence

$$(17) \quad b^3 - 2a^2b - a^3 - a - b = 0.$$

Using (17) in (14) we obtain $E = 0$, and so (9a) is proven. $\square$

3.3 Remark. The number $n = 2F_{2k}F_{2k+1}F_{2k+2}$ is divisible by 10. Recall that $F_m | F_n$ if and only if $m | n$ [2]. Both F_{2k} and F_{2k+2} are divisible by 2 so $4 | n$. Of three consecutive integers, one of them is divisible by 3. Thus, $10 | n$.

The triangular and Fibonacci families have some overlap. Observe that the first example is

$$(18a) \quad N_{10} = \mathcal{T}_{10} = \mathcal{F}_{10} = [[3, 4, 10, 11], [1, 10, 10, 15]].$$

Further observe that

$$(18b) \quad \mathcal{T}_{180} = [[13, 14, 180, 181], [3, 68, 180, 191]],$$

$$(18c) \quad \mathcal{F}_{180} = [[8, 26, 180, 182], [3, 68, 180, 191]],$$

and so

$$(18d) \quad N_{180} = [[13, 14, 180, 181], [8, 26, 180, 182], [3, 68, 180, 191]]$$

The sets (18a) and (18d) are the only overlaps between the two families. The overlap occurs whenever the equation

$$(19) \quad T(T+1) = 2F_{2k}F_{2k+1}F_{2k+2}$$

has a solution for some triangular number T . In fact, we have the following:

3.1 Theorem. $1 + 8F_{2k}F_{2k+1}F_{2k+2}$ is a perfect square only when $k = 0, 1, 2$.

Proof. By Cassini's identity we have $F_{2k}F_{2k+2} = F_{2k+1}^2 - 1$. Thus, let $x = F_{2k+1}$ so that $F_{2k}F_{2k+2} = x^2 - 1$, and we obtain the elliptic curve

$$y^2 = 8x^3 - 8x + 1.$$

A search for squares out to $x \leq 10^6$ yields Table 1.

Table 1: Integer points of the elliptic curve $y^3 = 8x^3 - 8x + 1$.

x	$8x^3 - 8x + 1 = y^2$
-1	1
0	1
1	1
2	$41 = 7^2$
5	$681 = 27^2$
6	$881 = 41^2$
7	$4701 = 75^2$
49	$585141 = 1217^2$
451	$498588781 = 22395^2$

The only Fibonacci numbers in the list are $x = F_{2k+1} \in \{0, 1, 2, 5\}$. Thus, keeping only $F_{2k+1} = 2, 5$ we obtain $T(T+1) = 10, 180$, and so $T = 3, 13$, respectively. $\square$

3.1 Conjecture. Suppose $|N_n| > 1$ for some positive integer n . Then $N_n = \mathcal{T}_n \cup \mathcal{F}_n$ and $|N_n| = |\mathcal{T}_n| + |\mathcal{F}_n| - |\mathcal{T}_n \cap \mathcal{F}_n|$.

4 The generalized Pell equation

Equation (2) is equivalent to the generalized Pell equation

$$(20) \quad x^2 - (d^2 + 1)y^2 = d^2,$$

with (d, y, dy, x) then being a solution to (2). We have the following statement on fundamental solutions.

4.1 Lemma (Fundamental solutions to the GPE (20)). 1. For any positive integer d , the fundamental matrix is

$$A_d = \begin{bmatrix} 2d^2 + 1 & 2d(d^2 + 1) \\ 2d & 2d^2 + 1 \end{bmatrix}. \text{ Observe that } |A_d| = 1.$$

2. For any positive integer d , $(x_0, e_0)_d = (d^2 - d + 1, \pm(d - 1))$.

3. For any positive integer d , $(x_0, e_0)_d = (d, 0)$. This solution is called the trivial solution and is discarded.

4. If $d = 2k^2$, then additionally we have the solutions $(x_0, e_0)_d = (2k^3 + k, \pm k)$.

5. If $d = F_{2k}$, then additionally we have the solutions $(x_0, e_0)_d = (L_{2k}, \pm 2)$, where L_{2k} is bisection of the Lucas sequence, S.6.11.

4.1 Remark. The author has been unable to show that these are all the fundamental solutions, but direct search supports the claim.

4.2 Remark. Observe that, as a consequence of the well-known identity $5F_r^2 + 4(-1)^r = L_r^2$ ([2], p75), we have

$$2^2 + F_{2k}^2 + (2F_{2k})^2 = L_{2k}^2,$$

yielding two pairs of solutions, namely,

$$[[7, \mathfrak{E}, 92, 93], [2, 47, 92, \mathfrak{Z}3]], \quad [[6, 7, 36, 37], [2, 19, 36, 3\mathfrak{E}]].$$

See rows 3–6 of Table 2 or Table 3.

4.3 Remark (Closing). The main reason the author thinks the two families are the only families is that when one iterates to a GPE solution $(x, e)_d$, considers all pairs in D_n^2 , then the only solutions to (2) found with $|N_n| > 1$ are in the two families.

5 Tables

The following tables are in duodecimal, sorted by $n = de$ and then by decreasing d .

Table 2: Solutions to $d^2 + e^2 + n^2 = x^2$ with $|N_n| > 1$ and $n \leq 10^4$.

	d^2	+	e^2	+	n^2	=	x^2
1.	3^2	+	4^2	+	10^2	=	11^2
2.	1^2	+	10^2	+	10^2	=	15^2
3.	6^2	+	7^2	+	36^2	=	37^2
4.	2^2	+	19^2	+	36^2	=	$3\mathfrak{E}^2$
5.	$\mathfrak{Z}^2$	+	$\mathfrak{E}^2$	+	92^2	=	93^2
6.	2^2	+	47^2	+	92^2	=	$\mathfrak{Z}3^2$
7.	13^2	+	14^2	+	180^2	=	181^2
8.	8^2	+	26^2	+	180^2	=	182^2
9.	3^2	+	68^2	+	180^2	=	191^2

□ Solutions to $d^2 + e^2 + n^2 = x^2$, $n \leq 10^4$.

10.	$19^2 + 17^2 + 326^2 = 327^2$
11.	$3^2 + 107^2 + 326^2 = 347^2$
12.	$24^2 + 25^2 + 578^2 = 579^2$
13.	$4^2 + 148^2 + 578^2 = 599^2$
14.	$30^2 + 31^2 + 930^2 = 931^2$
15.	$4^2 + 239^2 + 930^2 = 965^2$
16.	$39^2 + 37^2 + 1246^2 = 1247^2$
17.	$5^2 + 276^2 + 1246^2 = 1278^2$
18.	$47^2 + 48^2 + 1948^2 = 1949^2$
19.	$5^2 + 434^2 + 1948^2 = 1999^2$
20.	$19^2 + 154^2 + 2640^2 = 2645^2$
21.	$8^2 + 396^2 + 2640^2 = 2667^2$
22.	$56^2 + 57^2 + 2686^2 = 2687^2$
23.	$6^2 + 515^2 + 2686^2 = 2717^2$
24.	$66^2 + 67^2 + 3696^2 = 3697^2$
25.	$6^2 + 717^2 + 3696^2 = 3747^2$
26.	$77^2 + 78^2 + 4718^2 = 4719^2$
27.	$7^2 + 838^2 + 4718^2 = 4789^2$
28.	$89^2 + 87^2 + 6536^2 = 6537^2$
29.	$7^2 + 806^2 + 6536^2 = 6608^2$
30.	$70^2 + 71^2 + 8470^2 = 8471^2$
31.	$8^2 + 1073^2 + 8470^2 = 8575^2$
32.	$84^2 + 85^2 + 7948^2 = 7949^2$
33.	$8^2 + 1421^2 + 7948^2 = 7749^2$

Table 3: Solutions in the triangular family $\mathcal{T}$ from Table 2.

	$d^2 + e^2 + n^2 = x^2$
1.	$3^2 + 4^2 + 10^2 = 11^2$
2.	$1^2 + 10^2 + 10^2 = 15^2$
3.	$6^2 + 7^2 + 36^2 = 37^2$
4.	$2^2 + 19^2 + 36^2 = 38^2$
5.	$7^2 + 8^2 + 92^2 = 93^2$
6.	$2^2 + 47^2 + 92^2 = 73^2$
7.	$13^2 + 14^2 + 180^2 = 181^2$
8.	$3^2 + 68^2 + 180^2 = 191^2$
9.	$19^2 + 17^2 + 326^2 = 327^2$

$\square$ n in the triangular family, $n < 10^4$.

10.	$3^2 + 107^2 + 326^2 = 347^2$
11.	$24^2 + 25^2 + 578^2 = 579^2$
12.	$4^2 + 148^2 + 578^2 = 599^2$
13.	$30^2 + 31^2 + 930^2 = 931^2$
14.	$4^2 + 239^2 + 930^2 = 965^2$
15.	$39^2 + 37^2 + 1246^2 = 1247^2$
16.	$5^2 + 276^2 + 1246^2 = 1278^2$
17.	$47^2 + 48^2 + 1948^2 = 1949^2$
18.	$5^2 + 434^2 + 1948^2 = 1999^2$
19.	$56^2 + 57^2 + 2686^2 = 2687^2$
20.	$6^2 + 515^2 + 2686^2 = 2717^2$
21.	$66^2 + 67^2 + 3696^2 = 3697^2$
22.	$6^2 + 717^2 + 3696^2 = 3747^2$
23.	$77^2 + 78^2 + 4718^2 = 4719^2$
24.	$7^2 + 838^2 + 4718^2 = 4789^2$
25.	$89^2 + 87^2 + 6536^2 = 6537^2$
26.	$7^2 + 806^2 + 6536^2 = 6608^2$
27.	$70^2 + 71^2 + 8470^2 = 8471^2$
28.	$8^2 + 1073^2 + 8470^2 = 8575^2$
29.	$84^2 + 85^2 + 7948^2 = 7949^2$
30.	$8^2 + 1421^2 + 7948^2 = 7749^2$

Table 4: Solutions in the Fibonacci family $\mathcal{F}$ from Table 2.

	$d^2 + e^2 + n^2 = x^2$
1.	$3^2 + 4^2 + 10^2 = 11^2$
2.	$1^2 + 10^2 + 10^2 = 15^2$
3.	$8^2 + 26^2 + 180^2 = 182^2$
4.	$3^2 + 68^2 + 180^2 = 191^2$
5.	$19^2 + 154^2 + 2640^2 = 2645^2$
6.	$8^2 + 396^2 + 2640^2 = 2667^2$

6 Sequences in the OEIS

The definitions in the sequences have been modified to suit the needs of this document.

6.1 Sequence (A000027). The positive integers, $a(n) = n$.

6.2 Sequence (A000032). The Lucas sequence, $L(n) = F(n+1) + F(n-1)$, where $F(n)$ is the Fibonacci sequence. Also, $L(-n) = (-1)^{n+1}L(n)$.

6.3 Sequence (A000045). The Fibonacci sequence, $F(n) = F(n-1) + F(n-2)$ with $F(0) = 0$ and $F(1) = 1$. Also, $F(-n) = (-1)^{n+1}F(n)$.

6.4 Sequence (A000124). Central polygonal numbers: $a(n) = n(n+1)/2 + 1$.

6.5 Sequence (A000217). Triangular numbers: $a(n) = n(n+1)/2$. Equivalently, numbers T such that $1 + 8T = (2n+1)^2$ is a square.

6.6 Sequence (A000384). First hexagonal numbers: $a(n) = n(2n-1)$. Also the sequence of odd triangular numbers.

6.7 Sequence (A001621). $a(n) = T_n(T_n+1) = n(n+1)(n^2+n+2)/4$, where T_n is the sequence of triangular numbers.

6.8 Sequence (A001906). Bisection of Fibonacci sequence: $a(n) = F(2n)$.

6.9 Sequence (A002411). Pentagonal pyramidal numbers: $a(n) = nT_n = n^2(n+1)/2$.

6.10 Sequence (A004526). Nonnegative integers repeated: $a(n) = \text{floor}(n/2)$. Equivalently, $a(n) = (n+1)/2$ if n is odd, and $a(n) = n/2$ if n is even.

6.11 Sequence (A005248). Bisection of Lucas sequence: $a(n) = L(2n)$, where $L(n) = F(n+1) + F(n-1)$ and $F(n)$ is the Fibonacci sequence, S.6.3.

6.12 Sequence (A006000). $a(n) = (n+1)(T_n+1)$.

6.13 Sequence (A014105). Second hexagonal numbers: $a(n) = n(2n+1)$. Also the sequence of even triangular numbers.

6.14 Sequence (A240836). $a(n) = 2F_{2n-1}F_n$.

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