

A NOTE ON THE RAMIFICATION CHARACTER, RAMIFICATION POLYNOMIALS AND ASSOCIATED DIRICHLET L-FUNCTION

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ABSTRACT. In this paper, we introduce and study special types of polynomials whose coefficients are determined by the ramification characters $\kappa_m(\cdot)$ in a fixed modulo $m \geq 2$. These characters serve as indicator functions for integers that admit a binary representation where a summand is generated from the residue class of a modulus of lower magnitude by a given positive integer. We derive analytical properties for these polynomials investigating the Dirichlet L-function $L(s, \kappa_m(\cdot)) := \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s}$ associated with the ramification character.

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1. INTRODUCTION

Arithmetic information defined by congruence conditions can often be organized into sequences, generating functions, and polynomials. Such an encoding makes it possible to study the same arithmetic structure from both an algebraic and analytic point of view. The notion of ramification in a fixed modulus, developed in earlier work (see, e.g., [2]), provides a setting in which to pursue this approach. In this framework, an integer is a ramifier modulo a fixed integer when its residue modulo that integer is complementary to a residue attained modulo a smaller modulus. The resulting condition links congruence

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arithmetic with an additive relation between residue representatives.

Fix an integer modulus $m \geq 2$. We write $\kappa_m(n)$ for the indicator of ramification modulo m : thus $\kappa_m(n) = 1$ when n ramifies modulo m , and $\kappa_m(n) = 0$ otherwise. A key property used throughout the paper is the periodicity

$$\kappa_m(n + m!) = \kappa_m(n).$$

Consequently, the ramification character determines both finite polynomial encoding and an associated Dirichlet series. For a positive integer N , the ramification polynomial is

$$P_{N,m}(x) = \sum_{n=1}^N \kappa_m(n) x^n.$$

The purpose of this paper is to develop elementary algebraic properties of these polynomials, to provide criteria that help identify their nonzero coefficients, and to examine the analytic information carried by the periodic character. We write

$$L(s, \kappa_m) = \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s}$$

for the Dirichlet series associated with the ramification character. This notation records the Dirichlet-series construction; it does not require κ_m to be a classical Dirichlet character.

The use of polynomials to investigate arithmetic coefficient sequences has a substantial history. In particular, Fekete polynomials, whose coefficients are values of a quadratic character, provide a classical setting for the study of polynomial zeros and their relationship with associated Dirichlet L -functions. The work of Fekete and Pólya (see, e.g. [3]), and the later analysis of Conrey, Granville, Poonen, and Soundararajan (see, e.g. [1]), are relevant precedents for this general viewpoint. The ramification polynomials considered here have a different coefficient structure: their coefficients are indicator values in $\{0, 1\}$ rather than signed character values. The present paper therefore develops the corresponding generating-function framework for ramification, rather than assuming that zero results for Fekete polynomials carry over to this setting.

We first establish a convenient test for ramification in a fixed modulus. We then study the polynomials $P_{N,m}$, including their degree, a shift identity arising from the period $m!$, and the bounds on the closed unit disk. We also give a conditional product identity for polynomials associated with distinct moduli. Its hypothesis is expressed as a convolution identity for the coefficients, reflecting the fact that the coefficient of a product is determined by the number of pairs of exponents with a prescribed sum. A corresponding conditional addition identity is obtained from a coefficient-wise relation between the relevant ramification characters.

The arithmetic analysis is then refined by a test for ramification of a reduced residue representative. This criterion provides a more direct way to determine whether the selected coefficients are nonzero. The reduced ramification polynomial

$$P_m(x) = \sum_{k=1}^m \kappa_m(k) x^k$$

is considered separately. In particular, the paper expresses $P_m(1)$ as a count of indices satisfying an explicit inequality and uses this formula to show that $P_m(1) > 0$ for every

$m \geq 4$. Thus, $x = 1$ is not a zero of the reduced polynomial in that range.

The analytical part of the paper begins with a Mellin–gamma integral representation of $L(s, \kappa_m)$ for real $s > 1$. The periodicity of the character yields a second representation in terms of the one-period polynomial:

$$\Gamma(s)L(s, \kappa_m) = \int_0^1 \frac{(-\log x)^{s-1}}{x} \frac{P_{m!,m}(x)}{1-x^{m!}} dx, \quad s > 1.$$

The restriction $s > 1$ is the natural range for the arguments given here, since the defining Dirichlet series converges absolutely there and the integral is convergent at the endpoint $x = 1$. Finally, grouping the series into residue classes modulo $m!$ gives a local description near $s = 1$. If

$$A_m = \sum_{r=1}^{m!} \kappa_m(r),$$

then the associated Dirichlet series has a simple pole at $s = 1$ with residue $A_m/m!$. This identifies the leading singular behaviour in terms of the number of ramifiers in one period.

1.1. Organization of the paper. The paper is organized as follows. Section 2 recalls the ramification character and establishes a general ramification test. Section 3 introduces the fixed-modulus ramification polynomials and proves their elementary properties, including the conditional identities for products and sums. Section 4 develops the reduced residue-class test, and Section 5 studies the reduced ramification polynomial and its value at $x = 1$. Section 6 establishes the integral representations for the associated Dirichlet series and determines its pole and residue at $s = 1$. Section 7 concludes with remarks and open questions concerning the location of complex zeros of the ramification polynomials.

2. PRELIMINARY

In previous studies (see, e.g. [2]), we introduced the concept of ramification in a fixed modulus as a formal bridge between modular arithmetic and binary additive problems. Formally, an integer $n \geq 2$ with $n \equiv a_1 \pmod{m}$ is said to ramify in $\pmod{m}$ if there exists some integer r with $1 < r < m$ and $n \equiv a_2 \pmod{r}$ such that $a_1 + a_2 = m$. We say that the modulus m admits a ramifier and we denote the ramifier by $\mathcal{R}(m) = n$ as a shorthand. The following are some notable properties of ramifiers in a fixed modulus:

- There exists a ramifier in a fixed modulus.
- There are infinitely many ramifiers in a fixed modulus.
- Let $m \geq 2$ be a fixed integer. If $\mathcal{R}(m) = n$, then $\mathcal{R}(m) \not\equiv 0 \pmod{m}$.
- Let $m \geq 2$ be a fixed integer. We have

$$\#\{n \leq x : \mathcal{R}(m) = n\} \leq (1 - \frac{1}{m})x + O_m(1).$$

- There exists at least one integer $n \leq x$ that ramify in at least two moduli $m_1, m_2 \leq x$.

We define the ramification character on an integer $n \geq 2$ by

$$\kappa_m(n) := \begin{cases} 1 & \text{if } \mathcal{R}(m) = n \\ 0 & \text{otherwise.} \end{cases}$$

These characters serve as an indicator function on the set of all ramifiers in a fixed modulus. We check that for any fixed modulus $m \geq 2$, we have $\kappa_m(1) = 0$. The following properties of the ramification character (see, e.g., [2]) have been studied

- For a fixed modulus $m \geq 2$, we have $\kappa_m(n + m!) = \kappa_m(n)$.
- Let $m \geq 2$ be a fixed integer. We have $\kappa_m(n) = 0$ for $n \equiv 0, 1 \pmod{m}$.
- We have $\kappa_m(nm!) = \kappa_m(n)\kappa_m(m!)$.

Now, we fix the moduli $m_1 = 7$, $m_2 = 5$. We deduce $13 \equiv 6 \pmod{7}$ and $13 \equiv 1 \pmod{6}$. Because $6 + 1 = 7$, we conclude that $n = 13$ is a ramifier in $\pmod{7}$ and write $\mathcal{R}(7) = 13$. Similarly, we deduce $9 \equiv 4 \pmod{5}$ and $9 \equiv 1 \pmod{4}$. We check that $5 = 4 + 1$ and conclude that $n = 9$ is a ramifier in $\pmod{5}$.

The value(s) of the character $\kappa_m(\cdot) \in \{0, 1\}$ for a fixed modulus allows the construction of a class of polynomials whose coefficients depend on the integer arguments of the character and the fixed modulus $m \geq 2$. For a sufficiently large fixed modulus m , it may require some intensive arithmetic to check the ramification status of an arbitrary integer $n > m$. The following is a simple test that allows one to quickly decide on the ramification status of an arbitrary integer $n > m$.

Proposition 2.1 (The general ramification test). *Let $m \geq 2$ be a fixed integer and let $n \equiv a \pmod{m}$ be such that $n - m + a < m$. The integer n ramifies in $\pmod{m}$ (i.e., $\mathcal{R}(m) = n$) provided that the inequality $\frac{n+a}{2} < m < a + \frac{n}{2}$ is satisfied.*

Proof. Suppose that for a fixed $m \geq 2$, we have $n \equiv a \pmod{m}$. Furthermore, we write $n \equiv m - a \pmod{(n - m + a)}$. Hence, the integer n ramifies in $\pmod{m}$ provided that $n - m + a < m$ and $m - a < n - m + a$. This implies the inequality

$$\frac{n+a}{2} < m < a + \frac{n}{2}$$

must be satisfied. □

Example 2.2. We fix the modulus $m = 7$, choose the integer $n = 9$ and check that $n \equiv 2 \pmod{7}$ with $n - m + a < m$. However, the inequality $\frac{n+a}{2} < m < a + \frac{n}{2}$ fails. Thus, the integer $n = 9$ does not ramify in $\pmod{7}$, i.e., $\mathcal{R}(7) \neq 9$.

Example 2.3. We fix the modulus $m = 7$, choose the integer $n = 10$, and check that $n \equiv 3 \pmod{7}$ with $n - m + a < m$. Furthermore, the inequality $\frac{n+a}{2} < m < a + \frac{n}{2}$ is satisfied. Thus, the integer $n = 10$ does ramify in $\pmod{7}$, i.e., $\mathcal{R}(7) = 10$.

3. FIXED MODULUS RAMIFICATION POLYNOMIAL

This section formally introduces a class of polynomials whose coefficients are the values of the ramification characters $\kappa_m(\cdot)$ for a fixed integer modulus $m \geq 2$.

Definition 3.1. Let $m, n \geq 2$ be fixed integers. We put

$$P_{n,m}(x) := \sum_{i=1}^n \kappa_m(i)x^i$$

where

$$\kappa_m(i) = \begin{cases} 1 & \text{if } \mathcal{R}(m) = i \\ 0 & \text{otherwise} \end{cases}$$

is the ramification character in a fixed modulus m and $\mathcal{R}(m) = i$ means that i ramifies in modulo m . We call the polynomial $P_{n,m}(x)$ the ramification polynomial in a fixed modulus $m \geq 2$.

Example 3.2. We fix the modulus $m = 7$ and choose $n = 13$. We record the following values of the character $\kappa_7(13) = 0, \kappa_7(12) = 1, \kappa_7(11) = 1, \kappa_7(10) = 1, \kappa_7(9) = 0, \kappa_7(8) = 0, \kappa_7(7) = 0, \kappa_7(6) = 1, \kappa_7(5) = 1, \kappa_7(4) = 0, \kappa_7(3) = 0, \kappa_7(2) = 0, \kappa_7(1) = 0$. We now write the associated ramification polynomial

$$P_{13,7}(x) = x^{13} + x^{12} + x^{11} + x^{10} + x^6 + x^5.$$

The degree of this polynomial is $\max\{n \leq 13 : \mathcal{R}(7) = n\}$, i.e., the greatest ramifier in (mod 7) that does not exceed 13.

3.1. Elementary properties of the ramification polynomial. This section collects and investigates some elementary properties of the ramification polynomial in fixed moduli.

Proposition 3.3. *Let $m, n \geq 2$ be fixed integers. Denote $\deg(P_{n,m}(x))$ as the degree of the ramification polynomial $P_{n,m}(x)$. The following properties hold:*

(1) *We have $\deg(P_{n,m}(x)) = \max\{i \leq n : \mathcal{R}(m) = i\}$, i.e, the degree of any ramification polynomial $P_{n,m}(x)$ for a fixed $m, n \geq 2$ is always the largest ramifier not exceeding n .*

(2) *The representations $x^{m!}P_{n,m}(x) + P_{m!,m}(x) = P_{m!+n,m}(x)$ hold, i.e, ramification polynomials admit a quasi binary additive representation. In general, for any fixed $k \geq 1$, we have $x^{km!}P_{n,m}(x) + P_{km!,m}(x) = P_{n+km!,m}(x)$. In particular, $P_{n,m}(1) + P_{m!,m}(1) = P_{m!+n,m}(1)$.*

(3) *For all $|x| \leq 1$, we have*

$$|P_{n,m}(x)| \leq \#\{i \leq n : \mathcal{R}(m) = i\}$$

i.e, ramification polynomials are bounded in a unit disk by the cardinality of the set of all ramifiers not exceeding n for a fixed $n \geq 2$.

(4) *For all $|x| \leq 1$, we have $|P_{n+m!,m}(x)| \leq |P_{n,m}(x)| + |P_{m!,m}(x)|$, i.e, the triangle inequality is satisfied by ramification polynomials on the unit disk under a fixed modulus m .*

Proof. For (1), we observe that the degree of the ramification polynomial $P_{n,m}(x) := \sum_{i=1}^n \kappa_m(i)x^i$ is the greatest integer i with $1 \leq i \leq n$ such that $\kappa_m(i) \neq 0$. Hence

$$\deg(P_{n,m}(x)) = \max\{i \leq n : \mathcal{R}(m) = i\}.$$

For (2), we can write using the property $\kappa_m(i) = \kappa_m(i + m!)$

$$\begin{aligned} P_{n,m}(x) &= \sum_{i=1}^n \kappa_m(i)x^i \\ &= \sum_{i=1}^n \kappa_m(i + m!)x^i \\ (3.1) \quad &= \frac{1}{x^{m!}} \sum_{i=m!+1}^{n+m!} \kappa_m(i)x^i \end{aligned}$$

by re-indexing. By (3.1), we write

$$x^{m!}P_{n,m}(x) = \sum_{i=m!+1}^{n+m!} \kappa_m(i)x^i = P_{m,n+m!}(x) - P_{m,m!}(x).$$

Hence, we deduce the claimed additive representation. The general version $x^{km!}P_{n,m}(x) + P_{km!,m}(x) = P_{n+km!,m}(x)$ for any fixed $k \geq 1$ can also be deduced in a similar way. Therefore, we deduce $P_{n,m}(1) + P_{m!,m}(1) = P_{n+m!,m}(1)$. For (3), we observe via the triangle inequality that if $|x| \leq 1$, then

$$\begin{aligned} |P_{n,m}(x)| &= \left| \sum_{i=1}^n \kappa_m(i)x^i \right| \\ &\leq \sum_{i=1}^n \kappa_m(i) \\ &= \#\{i \leq n : \mathcal{R}(m) = i\}. \end{aligned}$$

For (4), we deduce on the unit disk $|x| \leq 1$ the inequality $|P_{n+m!,m}(x)| = |P_{n,m}(x) + P_{m!,m}(x)| \leq |P_{n,m}(x)| + |P_{m!,m}(x)|$, establishing a triangle inequality on the unit disk. $\square$

Proposition 3.4 (Closure under distinct moduli). *Let $m_1, m_2, n \geq 2$ be fixed positive integers. Let $P_{n,m_1}(x), P_{n,m_2}(x)$ be ramification polynomials in fixed moduli m_1, m_2 . We have $P_{n,m_1}(x)P_{n,m_2}(x) = P_{2n,m_1m_2}(x)$ provided that*

$$\sum_{i=\max(1,k-n)}^{\min(n,k-1)} \kappa_{m_1}(i)\kappa_{m_2}(k-i) = \kappa_{m_1m_2}(k)$$

for every $2 \leq k \leq 2n$, which means that the existence of single instances of identical ramification status of complimentary pairs in distinct moduli corresponds to the ramification status of their complementary sum in combined moduli by product. Furthermore, we have $P_{n,m_1}(x) + P_{n,m_2}(x) = P_{n,m_1+m_2}(x)$ provided that

$$\kappa_{m_1}(i) + \kappa_{m_2}(i) = \kappa_{m_1+m_2}(i)$$

for each $1 \leq i \leq n$.

Proof. For fixed moduli $m_1, m_2 \geq 2$, provided that

$$\sum_{i=\max(1,k-n)}^{\min(n,k-1)} \kappa_{m_1}(i)\kappa_{m_2}(k-i) = \kappa_{m_1m_2}(k)$$

for every $2 \leq k \leq 2n$, which means that the existence of single instances of identical ramification status of complimentary pairs in distinct moduli corresponds to the ramification

status of their complementary sum in combined moduli by product, we write

$$\begin{aligned}
P_{n,m_1}(x)P_{n,m_2}(x) &= \left(\sum_{i=1}^n \kappa_{m_1}(i)x^i \right) \left(\sum_{j=1}^n \kappa_{m_2}(j)x^j \right) \\
&= \sum_{v=2}^{2n} \sum_{i=\max(1,v-n)}^{\min(n,v-1)} \kappa_{m_1}(i)\kappa_{m_2}(v-i)x^v \\
&= \sum_{v=1}^{2n} \kappa_{m_1 m_2}(v)x^v
\end{aligned}$$

where we have used $\kappa_{m_1 m_2}(1) = 0$. Thus, we deduce $P_{n,m_1}(x)P_{n,m_2}(x) = P_{2n,m_1 m_2}(x)$. Furthermore, we deduce

$$\begin{aligned}
P_{n,m_1}(x) + P_{n,m_2}(x) &= \sum_{i=1}^n \kappa_{m_1}(i)x^i + \sum_{i=1}^n \kappa_{m_2}(i)x^i \\
&= \sum_{i=1}^n (\kappa_{m_1}(i) + \kappa_{m_2}(i))x^i \\
&= \sum_{i=1}^n \kappa_{m_1+m_2}(i)x^i
\end{aligned}$$

provided that $\kappa_{m_1}(i) + \kappa_{m_2}(i) = \kappa_{m_1+m_2}(i)$ for each i with $1 \leq i \leq n$. $\square$

4. REDUCED RESIDUE CLASS RAMIFICATION IN FIXED MODULUS

We begin this section with the introduction of a simple criterion to determine the status of ramification of the integers in a reduced residue class of a fixed modulus $m \geq 2$.

Proposition 4.1 (Reduced residue class ramification test). *Let $m \geq 2$ be a fixed integer and $n \equiv a \pmod{m}$ such that $\mathcal{R}(m) = n$. We have $\mathcal{R}(m) = a$ if and only if $2a - m > 0$ with $a > 2a - m > \frac{a}{2}$. In particular, the integer a ramifies in modulo $m \geq 2$ if and only if $2a - m > 0$ with $a > 2a - m > \frac{a}{2}$.*

Proof. We observe that $a \equiv a \pmod{m}$ since $\mathcal{R}(m) = n$ and write $a \equiv m-a \pmod{(2a-m)}$. Thus, $\mathcal{R}(m) = a$ if and only if $2a - m > 0$ with $m - a < 2a - m < m$. The inequality holds for $a < m < \frac{3}{2}a$, which implies that $\mathcal{R}(m) = a$ if and only if $a > 2a - m > \frac{a}{2}$. $\square$

Example 4.2. We fix the modulus $m = 7$ and choose $n = 13$. We observe $13 \equiv 6 \pmod{7}$ where we take $a = 6$ and $m = 7$. We can check that the inequality $a > 2a - m > \frac{a}{2}$ is satisfied in this case. Hence, $\mathcal{R}(7) = 6$, which means that $a = 6$ is a ramifier in $(\pmod{7})$. Indeed, we verify $6 \equiv 6 \pmod{7}$ and $6 \equiv 1 \pmod{5}$.

Example 4.3. We fix the modulus $m = 7$ and choose $n = 18$. We observe $18 \equiv 4 \pmod{7}$ where we take $a = 4$ and $m = 7$. We can check that the inequality $a > 2a - m > \frac{a}{2}$ is not satisfied in this case. Hence, $\mathcal{R}(7) \neq 4$, which means that $a = 4$ is not a ramifier in $(\pmod{7})$. Indeed, we verify $4 \equiv 4 \pmod{7}$, $4 \equiv 4 \pmod{6}$, $4 \equiv 4 \pmod{5}$, $4 \equiv 0 \pmod{4}$, $4 \equiv 1 \pmod{3}$, $4 \equiv 0 \pmod{2}$.

The introduced ramification test can be used to construct a relaxed version of ramification polynomials for large fixed modulus m . In principle, the simple test allows to easily detect

the coefficient of each term in the polynomial without going through the manual brute-force checking for the status of ramification via the character $\kappa_m(\cdot)$ for each $n \leq m$ for sufficiently large modulus m .

5. REDUCED RAMIFICATION POLYNOMIALS IN A FIXED MODULUS

This section considers a simpler version of the ramification polynomial and investigates the existence of zeros on the boundary of the half-open interval $(0, 1]$ of real numbers.

Definition 5.1. Let $m \geq 2$ be a fixed positive integer. We set

$$P_{m,m}(x) = P_m(x) := \sum_{i=1}^m \kappa_m(i)x^i$$

and call it the *reduced* ramification polynomial modulo $m \geq 2$.

Theorem 5.2 (A local boundary zero-free theorem). *Let $m \geq 2$ be a fixed integer. We have*

$$P_m(1) = \sum_{k=1}^m \kappa_m(k) = \# \left\{ m - k : m - 2k \geq 2, 1 \leq k < \frac{m}{3} \right\}.$$

In particular, the ramification polynomial $P_m(x) := \sum_{k=1}^m \kappa_m(k)x^k$ has no zeros at $x = 1$ for a fixed modulus $m \geq 4$.

Proof. We observe that for a fixed modulus $m \geq 2$, we have $m - k \equiv m - k \pmod{m}$ for integers $1 \leq k < m$. Furthermore, we obtain $m - k \equiv k \pmod{(m - 2k)}$ and $\mathcal{R}(m) = m - k$ if and only if $m - 2k \geq 2$ with $1 \leq k < \frac{m}{3}$. Therefore, we have the following relation

$$P_m(1) = \sum_{k=1}^m \kappa_m(k) = \# \left\{ m - k : m - 2k \geq 2, 1 \leq k < \frac{m}{3} \right\}.$$

We observe that for $m \geq 4$ the inequality $m - 2 \geq m - 2k > \frac{m}{3}$ holds and deduce

$$P_m(1) = \sum_{k=1}^m \kappa_m(k) = \# \left\{ m - k : m - 2k \geq 2, 1 \leq k < \frac{m}{3} \right\} > 0$$

implying that $x = 1$ cannot be a zero of the ramification polynomial $P_m(x) := \sum_{k=1}^m \kappa_m(k)x^k$ for all fixed modulus $m \geq 4$. $\square$

6. ANALYTIC PROPERTIES OF THE DIRICHLET L-FUNCTION ASSOCIATED WITH A RAMIFICATION CHARACTER

This section collects and investigates some analytic properties of the Dirichlet L-function associated with the ramification character $\kappa_m(\cdot)$ in a fixed modulus m . In this section, we will assume that for the fixed modulus m , the ramification character is not identically zero; in other words, we will assume that $\kappa_m(n) \neq 0$ for at least one n in a choice of domain.

Definition 6.1. Let $m \geq 2$ be a fixed integer. We set

$$L(s, \kappa_m(\cdot)) := \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s}$$

and call $L(s, \kappa_m(\cdot))$ the Dirichlet L -function associated with the ramification character $\kappa_m(\cdot)$.

Theorem 6.2 (Limiting integral representation). *Let $m \geq 2$ be a fixed integer that admits at least a ramifier in $(\text{mod } m)$. Denote the Dirichlet series associated with the ramification character $\kappa_m(\cdot)$ by*

$$L(s, \kappa_m(\cdot)) := \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s}.$$

For all $s > 1$, we have the integral representation

$$\Gamma(s)L(s, \kappa_m(\cdot)) = \int_0^1 \frac{(-\log x)^{s-1}}{x} \lim_{r \rightarrow \infty} \left(\frac{P_{r,m}(x)}{1-x^r} \right) \quad (0 < x < 1).$$

Proof. We recall the following identities related to the Euler Gamma function

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx = n^s \int_0^{\infty} x^{s-1} e^{-nx} dx = n^s \int_0^1 (-\log x)^{s-1} x^{n-1} dx.$$

Hence, we can write

$$\begin{aligned} \Gamma(s)L(s, \kappa_m(\cdot)) &= \Gamma(s) \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} \\ &= n^s \int_0^1 (-\log x)^{s-1} x^{n-1} \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} dx \\ &= \int_0^1 (-\log x)^{s-1} x^{n-1} n^s \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} dx \\ &= \int_0^1 \frac{(-\log x)^{s-1}}{x} \sum_{n=1}^{\infty} \kappa_m(n) x^n \\ &= \int_0^1 \frac{(-\log x)^{s-1}}{x} \lim_{r \rightarrow \infty} \left(\frac{P_{r,m}(x)}{1-x^r} \right) \quad (0 < x < 1). \end{aligned}$$

□

Theorem 6.3 (The local integral representation). *Let $m \geq 2$ be a fixed integer that admits at least a ramifier in $(\text{mod } m)$. Denote the Dirichlet series associated with the ramification character $\kappa_m(\cdot)$ by*

$$L(s, \kappa_m(\cdot)) := \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s}.$$

For all $s > 1$, we have the integral representation

$$\Gamma(s)L(s, \kappa_m(\cdot)) = \int_0^1 \frac{(-\log x)^{s-1}}{x} \frac{P_{m!,m}(x)}{1-x^{m!}} \quad (0 < x < 1).$$

Proof. We recall the following identities related to the Euler Gamma function

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx = n^s \int_0^{\infty} x^{s-1} e^{-nx} dx = n^s \int_0^1 (-\log x)^{s-1} x^{n-1} dx.$$

Hence, we can write

$$\begin{aligned}
\Gamma(s)L(s, \kappa_m(\cdot)) &= \Gamma(s) \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} \\
&= n^s \int_0^1 (-\log x)^{s-1} x^{n-1} \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} dx \\
&= \int_0^1 (-\log x)^{s-1} x^{n-1} n^s \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} dx \\
&= \int_0^1 \frac{(-\log x)^{s-1}}{x} \sum_{n=1}^{\infty} \kappa_m(n) x^n \\
(6.1) \quad &= \int_0^1 \frac{(-\log x)^{s-1}}{x} \frac{P_{m!,m}(x)}{1-x^{m!}} \quad (0 < x < 1)
\end{aligned}$$

where (6.1) is deduced using the property $\kappa_m(i + km!) = \kappa_m(i)$ for all $k \geq 1$. $\square$

Theorem 6.4 (The local profile theorem). *Let $m \geq 2$ be a fixed modulus that admits a ramifier in $(\text{mod } m)$. Denote by*

$$A_m := \sum_{i=1}^{m!} \kappa_m(i)$$

and

$$L(s, \kappa_m(\cdot)) = \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s}$$

as the Dirichlet L-function associated with the ramification character $\kappa_m(\cdot)$. The function $L(s, \kappa_m(\cdot))$ has a simple pole at $s = 1$ with residue $\frac{A_m}{m!}$. Equivalently,

$$L(s, \kappa_m(\cdot)) \sim \frac{A_m}{(s-1)m!} \quad (s \rightarrow 1^+).$$

Proof. We first assemble each term in the Dirichlet L-function associated with the ramification character $\kappa_m(\cdot)$ into residue classes $(\text{mod } m!)$ and use the periodicity property of the ramification character $\kappa_m(r + m!t) = \kappa_m(r)$, including the fact that $\kappa_m(0) = 0$ to

obtain

$$\begin{aligned}
L(s, \kappa_m(\cdot)) &= \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} \\
&= \sum_{t=0}^{\infty} \sum_{r=1}^{m!} \frac{\kappa_m(r + m!t)}{(r + m!t)^s} \\
&= \sum_{t=0}^{\infty} \sum_{r=1}^{m!} \frac{\kappa_m(r)}{(t + \frac{r}{m!})^s (m!)^s} \\
&= \sum_{r=1}^{m!} \frac{\kappa_m(r)}{(m!)^s} \sum_{t=0}^{\infty} \frac{1}{(t + \frac{r}{m!})^s} \\
(6.2) \quad &= \sum_{r=1}^{m!} \frac{\kappa_m(r)}{(m!)^s} \zeta(t, \frac{r}{m!})
\end{aligned}$$

where $\zeta(t, \frac{r}{m!}) := \sum_{t=0}^{\infty} \frac{1}{(t + \frac{r}{m!})^s}$ is the Hurwitz zeta function. The Hurwitz zeta function is known to have a simple pole at $t = 1$, which implies the residue at $t = 1$

$$\text{Res}_{s=1} \left(\sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} \right) = \frac{1}{m!} \sum_{r=1}^{m!} \kappa_m(r).$$

We conclude that

$$\sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s} \sim \frac{1}{m!(s-1)} \sum_{r=1}^{m!} \kappa_m(r) \quad (s \rightarrow 1^+).$$

□

7. FURTHER REMARKS AND AN OPEN PROBLEM

The current investigation introduces the concept of ramification polynomials as a special class of polynomials

$$P_{n,m}(x) = \sum_{i=1}^n \kappa_m(i) x^i$$

where

$$\kappa_m(i) = \begin{cases} 1 & \text{if } \mathcal{R}(m) = i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have shown a striking connection between this class of polynomials and the associated Dirichlet L -function $L(s, \kappa_m(\cdot))$ of the ramification character defined by

$$L(s, \kappa_m(\cdot)) := \sum_{n=1}^{\infty} \frac{\kappa_m(n)}{n^s}.$$

We pose the following problems for further investigation of the developed framework.

Problem 7.1. Let $m \geq 2$ be a fixed integer. Locate all the zeros of the ramification polynomial $P_{n,m}(x)$ for arbitrary n .

Problem 7.2. Fix a modulus $m \geq 2$. Does there exists a complex zero(s) with the real parts in the open interval $(0, 1)$ for the ramification polynomials $P_{n,m}(x)$, $P_m(x)$, $P_{m!,m}(x)$?

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