

Methodology of Studying the Two-Body Problem through Ellipse Kinematics: An Interdisciplinary Approach in Teaching Theoretical Mechanics

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Abstract

The article presents a methodology for studying the two-body problem and the law of universal gravitation through a kinematic analysis of elliptical motion. The proposed approach complements the traditional dynamical derivation of Kepler's laws with an inverse analysis: from the geometry of the trajectory to the structure of acceleration and its reconciliation with dynamical laws. This methodology enables students to gain a deeper understanding of the interrelation between kinematics and dynamics, develop mathematical thinking, and overcome conceptual difficulties in learning celestial mechanics. Typical student difficulties (the problem of mass, the illusion of contradiction between Newton's laws) and ways to overcome them are discussed. In addition to the first version, the article now contains an explicit algorithm for computing masses of bodies from observational data, using only kinematics and the experimental value of the gravitational constant G , without invoking the law of universal gravitation as a dynamical postulate. The proposed approach can be used in courses on theoretical mechanics, celestial mechanics, and general physics. The analytical results are additionally verified by numerical modelling on a mechanical ellipsograph, which confirms that the constancy of areal velocity is achieved only under a specific Keplerian law of angular velocity variation, not under arbitrary motion along an ellipse.

Keywords: teaching methods in physics, theoretical mechanics, kinematics of elliptical motion, two-body problem, Kepler's laws, interdisciplinary connections, Binet formula, typical learning difficulties, mass computation.

1. Introduction: Problematic Aspects of Teaching Celestial Mechanics

The study of Kepler's laws and the law of universal gravitation traditionally occupies a central place in courses of theoretical mechanics and general physics. However, teaching practice shows that students face a number of conceptual difficulties.

1.1. Formal Perception of Laws

Students often perceive physical laws as "given from above" postulates, without seeing their internal connection with the mathematical structure of the phenomena being described. Kepler's laws are studied as empirical facts, while the law of gravitation is treated as a dynamical postulate from which these laws are derived. Such an approach does not reveal the depth of the mathematical structure of celestial mechanics.

1.2. Gap between Mathematics and Physics

Students experience difficulties in understanding how the geometric properties of the trajectory (ellipse, focus) are related to physical laws (centrality of force, inverse-square law).

The mathematical apparatus (polar coordinates, differential equations) is perceived as a formal procedure rather than as a tool for analysing physical reality.

1.3. Conceptual Difficulties

Particular difficulties arise with:

- understanding why the centre of force lies exactly at the focus of the ellipse and not at its geometric centre;
- realising that from the kinematics of a single body one cannot obtain information about masses;
- overcoming the “illusion of contradiction” between Newton’s second law and the law of gravitation (why does the magnitude of velocity change if the force is directed towards the centre?);
- understanding the transition from the one-body problem to the two-body problem.

1.4. Aim of the Proposed Methodology

The proposed methodology addresses these problems through an inverse analysis: instead of the traditional derivation of kinematics from dynamics (Newton’s laws → Kepler’s laws), we conduct an analysis from kinematics to dynamics (geometry of the ellipse → structure of acceleration → reconciliation with Newton’s laws).

This approach:

- shows students how the mathematical structure of the geometry predetermines the physical structure of the field;
- demonstrates the uniqueness of the focus as the centre of force;
- explicitly separates kinematic relations from dynamical postulates;
- helps overcome conceptual difficulties through detailed discussion.

Important note: All mathematical results obtained within the proposed methodology are well known and presented in classical textbooks (Sivukhin, Whittaker, Goldstein). The novelty of the proposed approach lies not in obtaining new results, but in the methodological presentation that allows students to understand more deeply the logical foundations of mechanics.

1.5. To test the universality of the obtained relations, we also perform numerical modelling based on the classical Archimedes ellipsograph. By comparing different laws of angular velocity variation of the ruler, we show that Kepler’s second law is not a consequence of elliptical geometry alone – it requires a strictly defined kinematic regime, which was used as the basis of our derivation. This additionally substantiates the uniqueness of the focus as the centre of force and the inverse-square structure of the field.

2. Historical Context: From Observations to Mathematical Structure

Before proceeding to formal derivations, it is useful to reconstruct the historical logic of discoveries – it is itself an excellent illustration of how mathematics “grows” through empirical data.

2.1. Tycho Brahe and Data Accumulation

At the end of the 16th century, Tycho Brahe carried out uniquely accurate astronomical observations of the position of Mars. His data, obtained without a telescope, had an error of only about 2 arcminutes – small enough to distinguish between circular and elliptical orbital models.

2.2. Kepler: Empirical Laws

Johannes Kepler, analysing Tycho Brahe's data, formulated three laws (1609–1619):

1. First law: the orbits of planets are ellipses with the Sun at one focus.
2. Second law: the radius-vector of a planet sweeps out equal areas in equal times.
3. Third law: the square of the period is proportional to the cube of the semi-major axis.

It is important to emphasise to students: Kepler *did not know* why planets move this way. His laws were purely empirical – an accurate description of what happens, without an explanation of causes. This is a fine example of how mathematical description can precede physical understanding.

2.3. Newton: Synthesis and Explanation

Newton showed that all three Kepler's laws are consequences of a single law of gravitation and the second law of motion. However, in traditional exposition, this synthesis is often presented as purely deductive: “postulate the law of gravitation → derive Kepler's laws”. In doing so, an important methodological fact is lost: the reverse path – from kinematics to dynamics – is no less informative.

2.4. What Does Our Approach Offer?

By studying the inverse analysis (geometry → acceleration → force), students see that:

- the mathematical structure of the ellipse already “carries” the inverse-square dependence of acceleration on distance;
- Kepler's laws are not accidental empirical rules, but geometric properties of motion in a central field;
- the transition to the two-body problem “forces” the introduction of the product of masses – this is not an arbitrary postulate, but a mathematical necessity.

Thus, we not only explain how celestial mechanics works, but also show how mathematics and physics interpenetrate each other.

3. Kepler's Laws as Properties of the Ellipse

One of the key methodological emphases of the proposed approach is to present Kepler's laws not as external physical facts, but as *intrinsic geometric properties of elliptical motion* under certain conditions. Let us consider them in turn.

3.1. First Law as a Definition of the Trajectory

Kepler's first law states that the orbit of a planet is an ellipse with the central body at one focus. From the viewpoint of kinematic analysis, this is not a “physical explanation” law, but a choice of the class of trajectories from which we start our investigation. In our methodology, we take the ellipse as given (equation (1)) and investigate what follows from it. This allows

students to realise that the shape of the trajectory is not just an observed fact, but an underlying geometric structure that “dictates” the structure of acceleration.

3.2. Second Law as a Consequence of Centrality

As shown in Section 4, Kepler’s second law ($r^2 \cdot \dot{\varphi} = h = \text{const}$) arises automatically as soon as we require the acceleration to be directed towards the origin (the focus). But why do we require centrality? From a kinematic viewpoint – because only under central acceleration can the trajectory be an ellipse with constant parameters. In other words, Kepler’s second law is not an additional assumption, but a necessary condition for an elliptical trajectory to be realised in a central field at all. If the acceleration had a tangential component, the shape of the orbit would cease to be a stable ellipse.

3.3. Third Law as a Geometric Consequence

The third law ($T^2 \propto a^3$) is often perceived as an empirical rule connecting different planets. However, within our methodology it appears as a direct consequence of the fact that for any elliptical orbit in an inverse-square central field the quantity $\mu = \frac{4\pi^2 a^3}{T^2}$ is a constant for a given system. Geometrically, this means that the period and the semi-major axis are uniquely related – through the areal velocity $h = 2\pi ab/T$ and the focal parameter $p = \frac{b^2}{a} = a(1 - e^2)$. Since h and p are determined by the parameters of the ellipse, the third law is nothing but a relation between the geometric dimensions of the orbit and the time of its traversal.

Methodological conclusion: It is useful to offer students an independent exercise – to show that from the constancy of areal velocity and the equation of the ellipse follows $\frac{a^3}{T^2} = \text{const}$ without invoking any dynamical considerations. This exercise strengthens the understanding that Kepler’s laws are, essentially, “geometry in motion”, not arbitrary physical prescriptions.

4. Mathematical Apparatus: Kinematics of Motion along an Ellipse

Note: The complete derivation of the Binet formula and all intermediate transformations are given in Appendix A. In the main text, only the key steps and their physical interpretation are presented, so as not to overload junior students.

4.1. Statement of the Problem

Consider the motion of a material point along an elliptical trajectory. Place the origin of the polar coordinate system at one focus of the ellipse. The equation of the ellipse in polar coordinates is:

$$r = \frac{p}{1 + e \cdot \cos \varphi} \quad (1)$$

where $p = b^2/a = a(1 - e^2)$ is the focal parameter, e is the eccentricity, a and b are the semi-major and semi-minor axes.

Methodological comment: It is important to emphasise to students that the choice of the focus as the origin is not an arbitrary decision, but a mathematically justified step, which will be proved later (Section 5). The ellipse equation in this form is “tied” to the focus, and this is not accidental.

4.2. Derivation of Kepler’s Second Law from Kinematics

Switch to Cartesian coordinates:

$$x = r \cdot \cos(\varphi(t)) \quad (2)$$

$$y = r \cdot \sin(\varphi(t)) \quad (3)$$

Differentiating twice with respect to time, we obtain the components of acceleration. In polar coordinates, the radial and tangential components are:

$$w_r = \ddot{r} - r\dot{\varphi}^2, \quad w_\varphi = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\varphi}) \quad (4)$$

Key point: If we require the acceleration to be directed strictly towards the origin (the focus), the tangential component must vanish: $w_\varphi = 0$. This condition mathematically leads to:

$$\frac{d}{dt} (r^2 \dot{\varphi}) = 0 \implies r^2 \dot{\varphi} = h = \text{const} \quad (5)$$

which is exactly Kepler's second law (the law of areas).

Methodological comment: This derivation shows students that Kepler's second law is not merely an empirical fact, but a mathematical consequence of the requirement of central force. It is also important to note that if we chose the origin not at the focus but at any other point, this relation would cease to hold (see Section 5).

4.3. Kinematic Analogue of the Binet Formula

To determine the radial acceleration, we introduce the variable $u = 1/r$ and eliminate time using the area law. After transformations (full derivation in Appendix A) we obtain:

$$w_r = -h^2 \left(\frac{d^2 u}{d\varphi^2} + u \right) u^2 \quad (6)$$

Methodological comment: Formula (6) is obtained purely kinematically, without invoking any dynamical laws. This is an important pedagogical point: students see that the structure of acceleration is completely determined by the geometry of the trajectory.

4.4. Proof of the Inverse-Square Law

Substitute the ellipse equation (1) rewritten for the variable u :

$$u = \frac{1+e \cdot \cos \varphi}{p} \quad (7)$$

The second derivative is:

$$\frac{d^2 u}{d\varphi^2} = -\frac{e \cdot \cos \varphi}{p} \quad (8)$$

After substitution we get (see Appendix A for details):

$$w_r = -h^2 u^2 \left(-\frac{e \cdot \cos \varphi}{p} + \frac{1+e \cdot \cos \varphi}{p} \right) = -\frac{h^2}{p} \frac{1}{r^2} \quad (9)$$

Result: The acceleration is directed towards the focus and is inversely proportional to the square of the distance.

Methodological comment: This result demonstrates to students how the geometry of the ellipse mathematically predetermines the inverse-square field structure. Equation (9) is purely kinematic, but its form coincides with Newton's law of gravitation up to a factor.

5. Uniqueness of the Focus as the Centre of Force

5.1. Statement of the Problem

One of the most difficult conceptual questions for students: why is the centre of force located at the focus of the ellipse and not at its geometric centre?

5.2. Mathematical Proof of Uniqueness

Let us examine equation (6) more carefully. If we require the acceleration to depend on distance as $1/r^2$, then the expression in parentheses must be a constant:

$$\frac{d^2u}{d\varphi^2} + u = \text{const} \quad (10)$$

This is a linear second-order differential equation with constant coefficients. Its general solution is:

$$u = A + B \cos(\varphi - \varphi_0) \quad (11)$$

Geometrically, this equation describes a conic section (ellipse, hyperbola, or parabola), with the polar angle φ measured strictly from the focus of the curve.

5.3. Analysis of Alternative Cases

What happens if we choose a different point as the origin?

If we try to describe elliptical motion from the geometric centre of the ellipse (as shown by the author in a parallel study [5]), then:

- the areal velocity h ceases to be constant;
- the expression $u'' + u$ ceases to be constant;
- the law of force takes a linear form (Hooke's law $F \propto r$) rather than an inverse-square one.

Methodological comment: This analysis helps students understand that the focus is not an arbitrary choice, but the only geometric point with respect to which the kinematics of motion agrees with the Newtonian structure of the gravitational field. It is recommended to ask students, as an independent exercise, to derive the force law for the case when the origin is placed at the centre of the ellipse and compare the result with (9).

6. The Problem of Mass: From One Body to Two

6.1. Limitation of One-Body Kinematics

Conceptual problem: From the purely kinematic equations of motion of a single body along an ellipse, absolutely nothing can be said about its mass. This is a direct consequence of the equivalence principle: the trajectory in a given field does not depend on the mass of the test body.

Observing only one body, we can determine exclusively geometric and temporal parameters: the semi-major axis a , eccentricity e , period T , areal velocity h . We obtain the kinematic $\mu = \frac{4\pi^2 a^3}{T^2}$, but what this quantity actually is – from a purely kinematic point of view – we do not know.

Methodological comment: This point often causes difficulties. It is important to emphasise that kinematics describes how a body moves, but not why or what kind of body it is (in terms of mass). An analogy with a pendulum is useful: observing only its motion, we cannot determine its mass without knowing the spring stiffness.

6.2. Transition to the Two-Body Problem

The situation changes fundamentally when we move to the kinematics of two bodies observed simultaneously.

Consider a system of two bodies with masses M and m . Let r be the instantaneous distance between the bodies (between M and m) at a given moment. This is a variable quantity that changes from pericentre to apocentre.

Let a be the semi-major axis of their relative orbit. This is a constant geometric parameter of the ellipse along which one body moves relative to the other. In other words, a is half the maximum distance between the bodies (distance at pericentre plus distance at apocentre, divided by two).

When rotating around the common centre of mass (barycentre), the distances from each body to the centre of mass are divided in inverse proportion to their masses. Hence, for body m :

$$r_m = r \frac{M}{M+m}, \quad a_m = a \frac{M}{M+m}$$

where r_m is the instantaneous distance from body m to the barycentre, and a_m is the semi-major axis of its own orbit around the barycentre.

Apply the kinematic acceleration formula (9) we derived for body m , which orbits the barycentre with the same period T . The force acting on body m (by Newton's second $F = m \cdot a_m$) is:

$$F = m \cdot \frac{4\pi^2 a_m^3}{T^2} \cdot \frac{1}{r_m^2} = \frac{4\pi^2 a^3}{T^2} \cdot \frac{M}{(M+m)r^2} \quad (12)$$

6.3. Analysis of the Obtained Expression

Formula (12) is the key point where kinematics reconciles with dynamics:

1. Symmetry in masses. The kinematic interaction force is strictly proportional to the product of the masses $M \cdot m$. This is an algebraic consequence of both bodies orbiting the common barycentre with the same period T .
2. The proportionality coefficient depends only on the geometric and temporal parameters of the orbit (a and T) and on the sum of the masses.
3. Newton's third law confirms the symmetry of formula (12) with respect to masses M and m .

Important methodological clarification: The masses M and m appear in formula (12) not “from kinematics” but through the definition of the centre of mass $r_m = r \frac{M}{M+m}$, which already presupposes the existence of masses as physical quantities. Thus, we do not “derive” the dependence of force on masses, but rather show that, once the definition of the centre of mass and Newton’s second law are adopted, the algebraic structure of the expression for the force requires symmetry $F \propto M \cdot m$.

6.4. Hierarchy of Observations and Information about Masses

To systematise understanding, consider the hierarchy of observations:

Level of observation	What can be said about masses
Kinematics of one body	Nothing. Mass does not enter the equations.
Kinematics of two bodies (observing both orbits)	The ratio of masses can be found: $\frac{M_1}{M_2} = \frac{a_1}{a_2}$
Kinematics of two bodies + Newton’s second law	The mathematical structure of force requires it to be proportional to the product
Introduction of constant G	The sum of masses can be found: $M_1 + M_2 = \frac{4\pi^2 a^3}{GT^2}$

Methodological comment: This table (extended compared to the first version) helps students understand at which level of observation what information about masses becomes available. **Important:** to compute each mass individually, one needs the numerical value of G , which is taken from experiment (e.g., the Cavendish experiment). Kinematics alone gives only the mass ratio and their sum multiplied by G .

6.5. Computing Masses from Observational Data without Using the Law of Gravitation

We now come to the key methodological result that follows directly from the entire preceding analysis: **the masses of bodies can be computed from observational data without using the law of universal gravitation ($F = GMm/r^2$) as a dynamical postulate.**

For this, we need:

- Observed parameters of the motion of two bodies around the barycentre:
 - a_1 – semi-major axis of the orbit of body M_1 relative to the barycentre;
 - a_2 – semi-major axis of the orbit of body M_2 relative to the barycentre;
 - T – orbital period (same for both bodies).

The kinematic relation (Kepler’s third law as a property of the ellipse):

$$G(M_1 + M_2) = \frac{4\pi^2 a^3}{T^2}, \text{ где } a = a_1 + a_2$$

The mass ratio from the centre-of-mass definition (lever rule):

$$\frac{M_1}{M_2} = \frac{a_2}{a_1}$$

2. The experimental value of the gravitational constant G (e.g., $G = 6.674 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$).

Computation algorithm:

- **Step 1.** Find the relative semi-major axis: $a = a_1 + a_2$.
- **Step 2.** Find the sum of masses: $M_1 + M_2 = 4\pi^2 \cdot a^3 / (G \cdot T^2)$.
- **Step 3.** Find the mass ratio: $M_1/M_2 = a_2/a_1$.
- **Step 4.** Solve the system of two equations:
 $M_1 + M_2 = S$,
 $M_1/M_2 = a_2/a_1$.

We obtain:

$$M_1 = \frac{4\pi^2(a_1+a_2)^3}{GT^2} \cdot \frac{a_2}{a_1+a_2}, \quad M_2 = \frac{4\pi^2(a_1+a_2)^3}{GT^2} \cdot \frac{a_1}{a_1+a_2}$$

Important methodological note: Notice that the law of universal gravitation $F = GMm/r^2$ was not used anywhere in this computation. We used:

- kinematics of elliptical motion (formula (9) and Kepler's third law as its consequence);
- the definition of the centre of mass (lever rule);
- the experimental value of G , taken as an external fact.

The law of gravitation would be needed if we wanted to explain why bodies move along ellipses (dynamical cause), but for computing masses from observed orbits it is not required. This is an important pedagogical result that demonstrates to students that:

Kinematics + definition of centre of mass + experimental value of G are sufficient to determine masses. The law of universal gravitation as a dynamical postulate is not needed for this task.

This conclusion often surprises students who are used to thinking that masses “come from” the law of gravitation. In fact, the law of gravitation determines the force, while masses are determined from the motion.

Methodological comment: This section is useful to give as the final stage of studying the topic. It shows students the practical power of the kinematic approach and demonstrates that mass is not a “mystical” quantity, but a parameter extracted from observed motion.

6.6. Determination of the Gravitational Constant

Since for any planetary system the ratio a^3/T^2 is a constant, one can define the gravitational constant G as the coefficient that connects the kinematics of the orbit with the masses of the bodies:

$$G \equiv \frac{4\pi^2 a^3}{T^2(M+m)} \quad (13)$$

Methodological comment: It is important to emphasise that this is a definition of G , not its “derivation” from kinematics. The universality of G – i.e., the fact that the same constant

works for all systems – is an experimental fact that cannot be derived from pure kinematics. The numerical value and universality of G are established only by experiment. This definition is equivalent to the standard result following from Kepler’s third law in the Newtonian formulation for the two-body problem:

$$F = \frac{GMm}{r^2} \quad (14)$$

7. Typical Student Difficulties and Ways to Overcome Them

7.1. The “Illusion of Contradiction” between Newton’s Laws

Difficulty: If the force is directed strictly towards the focus, it has no tangential component; therefore the magnitude of the velocity should not change. However, in different points of the ellipse the speed is different.

Resolution of the contradiction: The cause of the apparent contradiction lies in the difference of the bases for decomposing acceleration:

- In the polar coordinate system, for a central force the azimuthal component $w_\phi = 0$, which leads to conservation of angular momentum.
- In the Frenet basis (tangent and normal), a radially directed force generally has a non-zero projection onto the direction of velocity (except at apsidal points). It is this projection that changes the magnitude of the velocity.

Methodological recommendation: Carry out a comparative analysis of the decomposition of acceleration in different bases, so that students can clearly see why the speed changes despite the centrality of the force.

7.2. The Question of the Uniqueness of the Focus

Difficulty: Why must the centre of force be exactly at the focus and not at the centre of the ellipse?

Resolution: The mathematical analysis of equation (6) gives an unambiguous answer: if $w_r \propto 1/r^2$, the trajectory equation has a solution only when the angle is measured from the focus. At the centre of the ellipse, the force law becomes linear (Hooke’s law).

Methodological recommendation: Ask students to derive the force law for the case when the origin is placed at the centre of the ellipse and compare the result with the inverse-square law.

7.3. The Problem of Mass in Kinematics

Difficulty: Why from the kinematics of a single body cannot one obtain information about its mass?

Resolution: This is a direct consequence of the equivalence principle. Mass “manifests” itself only in the transition to two bodies, when we observe the motion of both bodies around the common centre of mass.

Methodological recommendation: Use an analogy with a pendulum or with bodies falling in a gravitational field – acceleration does not depend on mass.

7.4. The 1.2% Discrepancy in the Earth–Moon System

Difficulty: In numerical calculations for the Earth–Moon system, a discrepancy of about 1.2% appears, which might be interpreted as a contradiction.

Resolution: This discrepancy refers exclusively to the simplified model (immovable central body, $M \rightarrow \infty$). In the full two-body model (taking into account the motion of the Earth around the barycentre), the kinematic and dynamical approaches give exactly identical results ($\delta = 0$).

The relative difference is:

$$\delta = \frac{M_2}{M_1} \approx \frac{1}{81.3} \approx 0.0123 = 1.23\%$$

Methodological recommendation: Use this example to discuss the importance of taking into account the motion of the central body and the effect of simplifications on accuracy.

7.5. (New) Why is the Law of Gravitation Not Needed for Computing Masses?

Difficulty: Students often think that masses “are determined” from the law of gravitation.

Resolution: As shown in Section 6.5, masses are computed from the kinematic parameters of the orbits and the experimental value of G . The law of gravitation ($F = GMm/r^2$) determines the force, but not the masses. Masses are inertial parameters that manifest themselves in motion (through the centre of mass and period). This is an important methodological distinction that helps students understand the structure of classical mechanics.

Methodological recommendation: After studying Section 6.5, ask students to compute the masses of the Earth and the Moon from real observational data using the given algorithm, and compare with tabulated values.

8. Numerical Example: Earth–Moon System

To confirm the consistency of the approaches, we calculate forces at the apsidal points.

Given:

- M_1 – mass of Earth;
- M_2 – mass of Moon;
- $e = 0.0549$ – orbital eccentricity;
- $T = 2,360,591.5$ s – orbital period;
- a – semi-major axis of the relative orbit.

Solution:

1. Kinematic accelerations in the inertial frame (relative to the barycentre).

Radii at pericentre $r_p = a(1 - e)$ and apocentre $r_a = a(1 + e)$.

According to formula (12), the inertial acceleration of body m (the Moon) in the field of body M (Earth) is:

$$W = \frac{4\pi^2 a^3}{T^2} \cdot \frac{M_1}{(M_1+M_2)r^2}$$

Then the accelerations at the apsidal points are:

$$w_p = \frac{4\pi^2 a^3}{T^2} \cdot \frac{M_1}{(M_1+M_2)} \cdot \frac{1}{r_p^2}$$

$$w_a = \frac{4\pi^2 a^3}{T^2} \cdot \frac{M_1}{(M_1+M_2)} \cdot \frac{1}{r_a^2}$$

2. Accelerations of the Earth's centre of mass.

From the mass ratio we find the semi-major axis of the Earth's orbit around the barycentre:

$$a_E = a \frac{M_2}{M_1+M_2}.$$

The acceleration of the Earth is computed analogously with a_E and the corresponding radius.

3. Calculation of forces by Newton's second law (kinematics).

Force acting on the Moon at perigee:

$$F_p = M_2 \cdot w_p = M_2 \cdot \frac{4\pi^2 a^3}{T^2} \cdot \frac{M_1}{(M_1+M_2)} \cdot \frac{1}{a^2(1-e)^2}$$

$$F_p = \frac{4\pi^2 a}{T^2} \cdot \frac{M_1 M_2}{(M_1+M_2)(1-e)^2}$$

Force at apogee:

$$F_a = M_2 \cdot w_a = M_2 \cdot \frac{4\pi^2 a^3}{T^2} \cdot \frac{M_1}{(M_1+M_2)} \cdot \frac{1}{a^2(1+e)^2}$$

$$F_a = \frac{4\pi^2 a}{T^2} \cdot \frac{M_1 M_2}{(M_1+M_2)(1+e)^2}$$

4. Calculation of forces by the law of universal gravitation.

Using formula (14), where G is defined by (13):

$$F = \frac{GMm}{r^2} = \left(\frac{4\pi^2 a^3}{T^2(M+m)} \right) \frac{M_1 M_2}{r^2}$$

Substituting the pericentre radius $r_p = a(1 - e)$:

$$F_p = \frac{4\pi^2 a}{T^2(M_1+M_2)} \cdot \frac{M_1 M_2}{a^2(1-e)^2} = \frac{4\pi^2 a}{T^2} \cdot \frac{M_1 M_2}{(M_1+M_2)(1-e)^2}$$

Similarly for apocentre $r_a = a(1 + e)$:

$$F_a = \frac{4\pi^2 a}{T^2(M_1+M_2)} \cdot \frac{M_1 M_2}{a^2(1+e)^2} = \frac{4\pi^2 a}{T^2} \cdot \frac{M_1 M_2}{(M_1+M_2)(1+e)^2}$$

Comparison of expressions: The expressions for F_p and F_a obtained in steps 3 and 4 are completely identical.

Result: relative discrepancy $\delta = 0\%$.

Methodological comment: This calculation clearly demonstrates that when using the full two-body model (taking into account the inertial acceleration relative to the barycentre), the kinematic and dynamical approaches give absolutely identical results. If, in step 3, one mistakenly uses the relative acceleration $\frac{4\pi^2 a^3}{T^2} \frac{1}{r^2}$ without the factor $\frac{M_1}{M_1 + M_2}$, one obtains exactly the 1.2% discrepancy discussed in Section 7.4.

9. Generalisation to Precessing Orbits

9.1. Statement of the Problem

Consider a generalisation to the case of precessing orbits (rosette orbits).

Define the trajectory as:

$$u = \frac{1}{p} (1 + e \cos(k\varphi)) \quad (15)$$

where k is the precession parameter ($k = 1$ – closed ellipse, $k \neq 1$ – precessing orbit).

9.2. Derivation of the Force Law

Substituting (15) into the Binet formula (6), we obtain:

$$w_r = -\frac{h^2}{pr^2} \left[1 + \frac{e}{p} (1 - k^2) \cos(k\varphi) \right] \quad (16)$$

9.3. Analysis of Special Cases

- When $k = 1$, we recover the classical inverse-square law.
- For small deviations $\delta = k - 1 \ll 1$:

$$w_r \approx -\frac{h^2}{pr^2} (1 - 2\delta \cos(k\varphi)) \quad (17)$$

Practical application. The kinematic formalism with the precession parameter k described above allows not only a theoretical description of rosette-shaped orbits, but also the direct extraction of the precession magnitude from observational data. In the author's work [7], a method is proposed based on the analysis of the difference of distances to the central body at symmetric (with respect to the line of apsides) points of the orbit. For one revolution, the function

$$D(\Delta t) = R_2(\Delta t) - R_1(\Delta t),$$

is constructed, where $R_1(\Delta t) = r(t_p + \Delta t)$ and $R_2(\Delta t) = r(t_p + T - \Delta t)$ are the distances at moments symmetric with respect to pericentre (t_p – time of pericentre passage, T – orbital period). A non-zero value of $D(\Delta t)$ is a direct sign of precession ($k \neq 1$), and its maximum or integral over the half-period allows one to quantitatively compute the angular shift of the pericentre per revolution $\Delta\varphi$:

$$\Delta\varphi \approx \frac{4D_{\max}}{a \cdot e} \text{ (approximate)}$$

or more accurately via the integral:

$$\Delta\varphi \approx \frac{4D_{\max}}{aeT} \int_0^{T/2} D(\Delta t) d(\Delta t).$$

This approach gives results consistent with classical secular measurements (e.g., for Mercury about 124" per century), but requires data from only one orbital period, making it convenient for educational and research tasks. A detailed description of the method, including the algorithm for working with NASA HORIZONS ephemerides, is given in [7].

9.4. Application to Real Systems

For the precession of the lunar perigee ($\Delta\varphi \approx 3^\circ$ per revolution), $k \approx 0.9916$, the correction is $\sim 1 - k^2 = 0.0167$. For Mercury, the precession is much smaller (43 arcseconds per century), which gives k even closer to unity. The main correction in (16) is due to external perturbations, not to a change in the structure of the central field.

Connection with perturbation theory: Within classical mechanics, orbital precession arises from additional forces (influence of other planets, non-sphericity of the central body). In GR, the precession of Mercury is explained by relativistic corrections to the metric. The kinematic formalism with parameter k allows one to describe such precession, but not to explain its cause – that is a task for dynamics.

Methodological recommendation: This section is useful for advanced students. It can be used to discuss the connection between kinematic description and perturbation theory, and to compare classical and relativistic mechanics using the example of Mercury's precession.

10. Limits of Applicability and Connection with Other Theories

10.1. Conditions of Applicability

The kinematic analysis is valid within the framework of classical mechanics under the following conditions:

1. Weak gravitational fields: $\frac{GM}{(rc^2)} \ll 1$, which excludes relativistic corrections.
2. Low velocities: $v \ll c$ (classical velocity addition).
3. Two bodies: strictly applicable only to the two-body problem (for N bodies it is approximate).
4. Moderate eccentricities: $0 \leq e < 1$ (any).

10.2. Connection with General Relativity

The law $F = \frac{GMm}{r^2}$ represents the Newtonian limit – an approximation valid for weak fields and low velocities. Relativistic corrections (precession of Mercury's perihelion) lie outside the scope of the considered kinematic analysis and require the use of the Schwarzschild metric.

Methodological comment: This point can be used to discuss the connection between classical and relativistic mechanics, showing students how Newtonian mechanics emerges as a limiting case of a more general theory.

11. Didactic Example for Independent Work

To consolidate the material, the following problem with a detailed solution is proposed.

Problem: Derive the force law for the case when the origin is placed at the geometric centre of the ellipse. Compare with the inverse-square law.

Brief solution:

The equation of the ellipse in polar coordinates with centre at the origin is:

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}$$

This expression does not have the form $r = p/(1 + e \cdot \cos \varphi)$ (unless $e = 0$). Substituting it into the Binet formula (6), we obtain:

$$w_r = -\frac{h^2}{r^2} \left(1 + \frac{3e^2}{1-e^2} \cos^2 \varphi + \dots \right)$$

For small e , this gives $w_r \propto \frac{1}{r^2}$ with corrections, but for $e = 0$ (circle) we get $w_r = \text{const} \cdot r$ – Hooke’s law. For $e \neq 0$, the law is not purely inverse-square; in particular, the areal velocity is not constant, and Kepler’s second law does not hold.

Methodological comment: This problem can be given in a seminar or as homework. It clearly demonstrates why the choice of the focus is essential and how geometry determines physics.

12. Methodological Recommendations for Teachers

12.1. Sequence of Study

1. Review of kinematics of curvilinear motion (polar coordinates, components of velocity and acceleration).
2. Historical excursion (Tycho Brahe → Kepler → Newton) – Section 2.
3. Kepler’s laws as properties of the ellipse – Section 3.
4. Kinematics of motion along an ellipse from the focus – Section 4 (main steps; full derivations in Appendix A).
5. Proof of uniqueness of the focus – Section 5.
6. The problem of mass and transition to two bodies – Section 6.
7. Analysis of typical difficulties – Section 7.
8. Numerical example – Section 8.
9. Generalisation to precessing orbits (for advanced students) – Section 9.
10. Limits of applicability and connection with GR – Section 10.
11. Didactic example – Section 11.

12.2. Forms of Organisation of Work

- **Lectures:** Theoretical material with emphasis on methodological aspects.
- **Seminars:** Solving problems, analysing typical difficulties, discussing conceptual questions.
- **Independent work:** Problems on deriving the force law for various trajectories, analysis of alternative cases.
- **Laboratory work:** Numerical simulation of elliptical motion, verification of Kepler’s laws.

12.3. Self-Assessment Questions and Problems

Questions for self-assessment:

1. Why is Kepler's second law satisfied only when the origin is placed at the focus?
2. What can be said about the mass of a body if we observe only its elliptical motion?
3. Why does the product of masses appear in the force formula when moving to the two-body problem?
4. How do you explain the change in the magnitude of velocity during elliptical motion if the force is directed towards the centre?
5. What are the limits of applicability of Newtonian mechanics?
6. In what sense are Kepler's laws properties of the ellipse?
7. Can masses of bodies be computed without using the law of universal gravitation? If yes, how?

Problems:

1. Derive the force law for the case when the origin is placed at the centre of the ellipse (using Section 11).
2. Calculate the forces at the apsidal points for the Earth–Moon system using both approaches.
3. Determine the precession parameter k for Mercury's orbit (from the known precession of 43" per century) and estimate the correction to the inverse-square law.
4. Show that from the constancy of areal velocity and the equation of the ellipse follows Kepler's third law.
5. Using real data for the Jupiter–Ganymede system ($(a_1 = 1.07 \times 10^9 \text{ m}, (a_2 = 0.3 \times 10^9 \text{ m}, T = 6.18 \times 10^5 \text{ s}, G = 6.674 \times 10^{-11})$), compute the masses of Jupiter and Ganymede.
6. **(Advanced task – research project)** Using open NASA HORIZONS ephemerides (<https://ssd.jpl.nasa.gov/horizons.cgi>) for any planet or Moon, construct the distance-difference function $D(\Delta t)$ for one orbital period according to the method described in [7]. To do this:
 - select a time interval equal to one orbital period of the chosen body and obtain a table of distances $r(t)$ with a step no larger than 1/100 of the period;
 - determine the pericentre passage time t_p ;
 - for each Δt from 0 to $T/2$, find $R_1 = r(t_p + \Delta t)$ and $R_2 = r(t_p + T - \Delta t)$, compute $D(\Delta t)$ and plot it;
 - using the maximum $D_{\max}$ or the integral, estimate the angular shift of the pericentre per revolution $\Delta\phi$ and convert it to arcseconds per century.

Compare your result with known tabulated data (e.g., for Mercury – 43" per century from GR, with all perturbations about 5600" per century). Explain why your estimate may differ from the classical relativistic value, and which factors (perturbations from other planets, non-sphericity of the Sun) contribute to the total precession. *Hint:* for numerical integration, the trapezoidal rule is convenient.

12.4. Interdisciplinary Connections

The proposed methodology establishes links between:

- Mathematics: differential equations, polar coordinates, theory of conic sections.
- Physics: kinematics, dynamics, gravitation, celestial mechanics.
- History of science: works of Kepler, Newton, development of mechanics.
- Philosophy of science: relationship between mathematics and physics, problem of uniqueness of solutions.

13. Numerical Verification on the Ellipsograph Model

In the previous sections, we rigorously derived that the inverse-square law and the constancy of areal velocity follow from the kinematics of elliptical motion **provided that** the angular motion of the point obeys a specific law derived from the geometry of the ellipse and conservation of angular momentum. However, an important question remains: is motion along an ellipse by itself sufficient for Kepler's second law, or does it require a specific kinematic organisation of velocity?

To answer this question, consider the classical mechanical Archimedes ellipsograph (Fig. 1). In this device, a point C on a ruler moving between two perpendicular guides describes an ellipse. The law of variation of the ruler's angle can be chosen arbitrarily – for example, uniform rotation, uniformly accelerated, or any other.

In the author's work [6], numerical modelling of three modes of ruler motion was performed:

1. Uniform angle change: $\alpha(t) = \omega_0 t$;
2. Uniformly accelerated change: $\alpha(t) = \frac{\varepsilon_0 t^2}{2}$;
3. Keplerian (elliptical) change, where the angular velocity obeys the condition of constant areal velocity relative to the focus of the ellipse [1].

For each mode, the areas of sectors swept by the radius-vector of point C over equal time intervals in different phases of motion were computed. The results are unambiguous:

- Under uniform and uniformly accelerated rotations of the ruler, the areal velocities are not constant – the areas of sectors taken over equal times differ significantly ($\Delta S \gg 0$). This means that Kepler's law of areas is not satisfied, even though the trajectory remains elliptical.
- Under the Keplerian mode, where the angle change is given by equation (13) – i.e., through the relation

$$\frac{a}{b} \tan \varphi = \tan \left(\frac{2\pi t}{T} \right)$$

– numerical integration gives strict equality of areas over any equal time intervals ($\Delta S \approx 0$ up to machine precision).

Thus, the ellipsograph model clearly demonstrates that **the elliptical shape of the trajectory by itself does not guarantee Kepler’s second law** – the necessary condition is a specific nonlinear law of angular velocity variation, which we used in the analytical derivation (Section 4). This result reinforces our approach: we do not postulate arbitrary motion along an ellipse, but single out the only kinematic regime consistent with Kepler’s laws, and show that it is from this regime that the Newtonian field structure follows.

This numerical experiment serves as an additional verification of the correctness of the transition from the kinematics of one body to the dynamics of two bodies. It is also useful pedagogically, allowing students to see the difference between “just an ellipse” and a “Keplerian ellipse”, and between centrally-symmetric description (Hooke’s law) and focus-oriented (Newtonian gravity).

14. Conclusion

The proposed methodology of studying the two-body problem through ellipse kinematics solves several pedagogical tasks:

1. Overcoming formal perception of laws. Students see how the mathematical structure of geometry predetermines the physical structure of the field, and they realise that Kepler’s laws are essentially properties of elliptical motion under certain conditions.
2. Development of mathematical thinking. The analysis of uniqueness of the focus, the transition from one body to two, and the generalisation to precessing orbits develop skills in mathematical analysis.
3. Overcoming conceptual difficulties. Detailed discussion of the mass problem, the “illusion of contradiction”, and the 1.2% discrepancy helps students understand the material more deeply.
4. Formation of an integral scientific worldview. Students see the connection between kinematics and dynamics, between classical and relativistic mechanics, and between the empirical discoveries of Kepler and the mathematical synthesis of Newton.
5. Demonstration of the practical power of the kinematic approach. Students learn that masses of bodies can be computed from observational data without using the law of universal gravitation as a dynamical postulate – kinematics, the definition of the centre of mass, and the

experimental value of G are sufficient. This is an important methodological lesson: the law of gravitation determines the forces, while masses are determined from the motion.

All mathematical results are well known; the novelty of the approach lies in the methodological presentation, which allows students to understand more deeply the logical foundations of mechanics. The proposed methodology can be used in courses on theoretical mechanics, celestial mechanics, and general physics as a supplement to the traditional dynamical approach.

References

1. Sivukhin D.V. *General Course of Physics*. In 5 vols. Vol. 1. Mechanics. – 4th ed. – Moscow: Fizmatlit, 2002. (in Russian)
2. Arnold, V. I. (1990). *Huygens and Barrow, Newton and Hooke*. Moscow: Nauka.
3. Newton, I. (1999). *The Principia: Mathematical Principles of Natural Philosophy*. University of California Press.
4. Whittaker, E. T. (1937). *Analytical Dynamics of Discrete Points and Rigid Bodies*. Moscow-Leningrad: ONTI. (in Russian)
5. Strohm, V. F. (2026). Differential Equation of Motion of a Material Point Along an Ellipse Relative to its Center (Variant 2). Manuscript. <https://zenodo.org/records/20546429>
6. Strohm, V. F. Kinematic Analysis of the Motion of an Ellipsograph Point under Various Velocity Laws. Zenodo, 2026. DOI: <https://doi.org/10.5281/zenodo.20557970>
7. Strohm, V. F. Mathematical Description of Oscillations of the Radius Difference Modulus at Symmetric Orbital Points. Zenodo, 2026. DOI: <https://doi.org/10.5281/zenodo.20067394>

Appendix A: Complete Derivation of the Binet Formula and the Inverse-Square Law

Here we give all intermediate derivations that were presented only as key steps in the main text.

A.1. Components of Velocity and Acceleration in Polar Coordinates

Let $x = r \cos \varphi$, $y = r \sin \varphi$. Then:

$$\dot{x} = \dot{r} \cos \varphi - r \sin \varphi \cdot \dot{\varphi}$$

$$\dot{y} = \dot{r} \sin \varphi + r \cos \varphi \cdot \dot{\varphi}$$

Second derivatives:

$$\ddot{x} = \dot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi$$

$$\ddot{y} = \dot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - r\dot{\varphi}^2\sin\varphi$$

From these, the radial and tangential components of acceleration are:

$$w_r = \ddot{r} - r\dot{\varphi}^2, \quad w_\varphi = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\varphi})$$

A.2. Transition to the Variable $u = 1/r$

We have:

$$\dot{r} = -\frac{\dot{u}}{u^2}, \quad \ddot{r} = -\frac{\ddot{u}}{u^2} + \frac{2\dot{u}^2}{u^3}$$

From the area law: $\dot{\varphi} = hu^2$.

Then:

$$\dot{r} = \frac{1}{u^2} \frac{du}{d\varphi} \dot{\varphi} = -h \frac{du}{d\varphi}$$

$$\ddot{r} = -h \frac{d^2u}{d\varphi^2} \dot{\varphi} = -h^2 u^2 \frac{d^2u}{d\varphi^2}$$

Radial acceleration:

$$w_r = \ddot{r} - r\dot{\varphi}^2 = -h^2 u^2 \frac{d^2u}{d\varphi^2} - \frac{1}{r} h^2 u^4 = -h^2 u^2 \left(\frac{d^2u}{d\varphi^2} + u \right)$$

which is formula (6).

A.3. Substitution of the Ellipse Equation

$$u = \frac{(1+\cos\varphi)}{p}$$

Then:

$$u'' + u = \frac{-e\cos\varphi + 1 + e\cos\varphi}{p} = \frac{h^2}{p} \frac{1}{r^2}$$

Hence:

$$w_r = -h^2 \cdot u^2 \cdot (1/p) = - (h^2/p) \cdot (1/r^2)$$

which coincides with (9).

Appendix B: Recommendations for Students

When studying this material, pay attention to:

1. The difference between kinematics and dynamics. Kinematics describes how a body moves; dynamics explains why it moves that way. Our approach shows how the kinematic structure agrees with dynamical laws.
2. Kepler's laws as properties of the ellipse. The first law defines the trajectory; the second is a consequence of centrality; the third is a consequence of geometric relations between a , b , T , and h . All three can be understood without invoking the law of gravitation.
3. The uniqueness of mathematical solutions. The focus of the ellipse is not an arbitrary choice, but the only point with respect to which Kepler's laws hold for an inverse-square field.
4. Hierarchy of information. At different levels of observation (one body, two bodies), different information about physical quantities (masses) becomes available.
5. Limits of applicability of models. Simplifications (immovable central body) lead to discrepancies that can be quantitatively estimated.
6. Connection with more general theories. Newtonian mechanics is a limiting case of a more general relativistic theory.

Recommended literature for in-depth study:

- Sivukhin D.V., *Mechanics*, sections on central field and the two-body problem.
- Whittaker E.T., *Analytical Dynamics*, chapters on the Kepler problem.
- Goldstein H., *Classical Mechanics*, section on the two-body problem.