The Tully-Fisher relation: an explanation based on general relativity

Fernando Salmon Iza
Physicist. Complutense University of Madrid.
Independent Researcher
fernandosalmoniza@gmail.com

Abstract

The empirical Tully-Fisher relationship for spiral galaxies has been the subject of study for many years. The flat character of the rotation speed curves of galaxies and the need for the existence of dark matter to explain them made it even more difficult to find a theoretical foundation between the fourth power of the rotation speed of stars and their luminous mass or baryonic mass, that is, without dark mass, which would serve to demonstrate the relationship that R. Brent Tully and J. Richard Fisher found in 1977 for spiral galaxies and which is very useful for calculating various astronomical calculations. was until now physically unprovable.

In this report we have demonstrated the Tolly-Fisher relationship using generalized relativity and applying it to a new energetic model of a galaxy. To achieve this, we use the Sérsic luminosity profiles as an estimate of baryonic mass and our new galaxy model to calculate the curvature of space-time within the galactic volume, subsequently deriving the acceleration experienced by the stars. The results show that this acceleration gives rise to a uniform velocity for each star, regardless of its position. as long as these are at a distance less than the effective radius. Furthermore, we have obtained a proportionality relationship between the luminosity of the galaxy and the fourth power of the rotation speed of the stars - the Tully-Fisher relationship - for spiral galaxies, which until now had been purely empirical.

Keywords: galaxy rotation curves, Tully-Fisher relation, general relativity, astrophysics

1.lntroduction

Since the 1930s it was known that the velocity of stars within a galaxy did not follow the pattern predicted by Newtonian gravitation. The work of Jan Hendrik Oort and Fritz Zwicky had shown that stellar velocities were greater than expected, which led them to propose the existence of a new, invisible mass of matter to explain this anomaly. However, it was not until the publication of astronomer Vera C. Rubín's research in 1980 (Rubín et al., 1980) that it was shown that the stars in a given galaxy maintain the same speed regardless of their position, meaning that the rotation curves are flat, contrary to the predictions of Newtonian gravitation.

The nature of the Tully-Fisher relationship is different: this empirical equation establishes a proportionality between the luminosity of a galaxy—or its baryonic mass—and the fourth power of the rotation speed of its stars. The relationship was published by R. Brent Tully and J. Richard Fisher in 1977 (Tully-Fisher, 1977) for spiral galaxies and constitutes a procedure for determining their distance. Until now, it lacked a theoretical basis; Its foundation was purely empirical. In this work, we have derived a theoretical basis based on special relativity and the galaxy model that we present in this report. Below, we briefly describe the method followed to reach these conclusions that led to the Tully-Fisher relationship

Our objective was to determine the curvature of spacetime within the volume of a galaxy, to subsequently calculate the accelerations and velocities of the stars.

To do this, we first establish the relationship between the baryonic mass and the luminosity profile for spiral-type galaxies, using Sérsic profiles (Sersic's law). Next, we calculate the curvature within a Schwarzschild spacetime, specifically the one defined by J. Droste (the Flamm paraboloid). Subsequently, we apply the Birkhoff-Jebsen theorem to determine the curvature of spacetime in an extended medium: our model galaxy, where the baryonic energy-matter density varies with position, based on baryonic mass data derived from Sérsic luminosity profiles. This result on curvature allows us, through the application of the Equivalence Principle, to calculate the relativistic accelerations experienced by the stars within this medium, demonstrating that all those that are at a distance less than the effective radius of the galaxy move at the same speed. In other words, the velocities of stars within a given spiral galaxy are identical, regardless of their position within the effective radius, without the need to make a dark matter hypothesis. Finally, we derive the Tully-Fisher relation as a proportionality between the luminosity of a galaxy or baryonic mass and the fourth power of the rotation speed of its stars, applying generalized relativity to our galaxy model, which represents an extended and continuous baryonic energy-matter medium of energy-matter densities obtained by studying the Sersic profile.

2. Distribution of baryonic mass using Sérsic profiles

The Sérsic profile (or Sérsic model or Sérsic law), (Sérsic J L, 1963), is a mathematical function that describes how the intensity I varies with the distance r from its center. It is a generalization of de Vaucouleurs' law. José Luis Sérsic first published his results in 1963. (Wikipedia 1, 2026).

The equation for the Sérsic profile is:

$I_r = I_e e^{-b_n((\frac{r}{r_e})^{\frac{1}{n}}-1))}$)

I_r = intensity at distance r from the center of the galaxy

I_0 = maximum intensity

I_e = effective intensity, $I_e = ½I_0$

r_e = effective radius corresponding to I_e

n is a value that depends on the type of galaxy., For spiral and exponential-type galaxies, such as the Milky Way, n = 1.

b_n is a value that depends on the type of galaxy. $b_n \cong 2n - (1/3)$.

The baryonic mass M of the galaxy will be proportional to the integral of its luminosity profile.

for spiral galaxies n = 1

$$M \sim \int_0^{\infty} I_e e^{-b_n(\left(\frac{r}{r_e}\right)-1)}\ d\left(\frac{r}{r_e}\right) =$$

$$= I_e \frac{e^{b^n}}{b_n} \sim I_e$$

The baryonic mass of the galaxy contained within a radius r, M(r), will be proportional to the integral of the luminosity L(r) from zero to r.

$e^x = 1 + x + x^2/2! + x^3/3! + ...$

$$M(r/r_e) \sim L\,(r/r_e) =$$

$$= \int_0^{\frac{r}{r_e}} I_e e^{-b_n(\left(\frac{r}{r_e}\right)-1)}\ d\left(\frac{r}{r_e}\right) \cong$$

$$\cong\ I_e \int_0^{\frac{r}{r_e}} (1 - b_n(\left(\frac{r}{r_e}\right) - 1))\ d\left(\frac{r}{r_e}\right)\ =$$

$$= I_e\ ((1 + b_n)\frac{r}{r_e}\ - ½\ b_n \left(\frac{r}{r_e}\right)^2)$$

for $0 < r < r_e$

$$L(r/r_e) \cong I_e\ (1 + b_n)\frac{r}{r_e} - \sim I_e \frac{r}{r_e}$$

$M(r/r_e)\ \sim L(r/r_e)\ \sim\ \frac{r}{r_e}$ So, for spiral galaxies the mass-luminosity relationship increases with the radius (Babcock H, 1939)

3.- Schwarzschild problem. Gaussian curvature of spacetime in the J. Droste solution.

Flamm's paraboloid. J. Droste's time-independent spacetime solution to the problem studied by Schwarzschild (Schwarzschild K, 1916) is a 2D surface. Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the point mass "r" and the azimuth angle "φ". The problem admits a mathematical treatment of differential geometry of surfaces [Wikipedia 2] 2026) and with it we are going to calculate the Gaussian Curvature. (R_s = Schwarzschild radius).

Equation of the Flamm paraboloid

Surface parameters (r, φ)

$0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi$

which has this parametric equation:

$x = r \cos\varphi$

$y = r \operatorname{sen}\varphi$

$z = 2(R_s (r- R_s))^{1/2}$

Vector Equation of the Flamm paraboloid

$f(x,y,z) = (r \cos\varphi, r \operatorname{sen}\varphi, 2(R_s (r- R_s))^{1/2})$

Determination of velocity, acceleration, and normal vectors to the surface

$\eth f/\eth\varphi = (-r \operatorname{sen}\varphi, r \cos\varphi, 0)$

$\eth f/\eth r = (\cos\varphi, \operatorname{sen}\varphi, (r/R_s -1)^{-1/2})$

$\eth^2 f/\eth\varphi^2 = (-r \cos\varphi, -r \operatorname{sen}\varphi, 0)$

$\eth^2 f/\eth r^2 = (0, 0, (-1/2R_s). (r/R_s -1)^{-3/2})$

$\eth f/\eth\varphi\eth r = (-\operatorname{sen}\varphi, \cos\varphi, 0)$

$$n = \frac{(\eth f/\eth\varphi \times \eth f/\eth r)}{[\frac{\eth f}{\eth\varphi} x \frac{\eth f}{\eth r}]}$$

$(\eth f/\eth\varphi \times \eth f/\eth r) =$

$= (r \cos\varphi/ (r/R_s -1)^{1/2}, r \operatorname{sen}\varphi/(r/R_s -1)^{1/2}, -r)$

$[\frac{\eth f}{\eth\varphi} x \frac{\eth f}{\eth r}] = r ((1/ (r/R_s -1)) +1)^{1/2}$

Curvature and curvature parameters

$K_{gauss} = LN-M^2/EG-F^2$

$L = \eth^2 f/\eth\varphi^2. \; n = -r(r/R_s)^{-1/2}$

$N = \eth^2 f/\eth r^2. \; n =$

$= (1/2R_s) (r/R_s)^{-1/2} (r/R_s - 1)^{-1}$

$M = (\eth f/\eth\varphi\eth r). \; n = 0$

$E = \eth f/\eth\varphi. \; \eth f/\eth\varphi = r^2$

$G = \eth f/\eth r. \; \eth f/\eth r = 1 + (1/ (r/R_s - 1))$

$F = \eth f/\eth\varphi. \; \eth f/\eth r = 0$

An equation of Gauss curvature

$K_{gauss} = LN-M^2/EG-F^2$

$K_{gauss} = -R_s/2r^3 = - GM/c^2r^3$

for Schwarzschild radius, $R_s = 2GM/c^2$ (Salmon Iza F, 2024)

4. The Birkhoff-Jebsen Theorem

We make a brief comment on this theorem of mathematics applied to the theory of generalized relativity [Weinberg S, 1972). First, we summarize Professor Fulvio Melia in reference [Melia F, 2022] to explain it According to the Birkhoff-Jebsen theorem, we can state that:

"If we have a spherical universe of mass-energy density ρ and radius r and within it a concentric sphere of radius r_s smaller than r, it is true that the acceleration due to gravity at any point on the surface of the sphere of relative radius r_s to an observer at its origin, depends solely on the mass-energy relation contained within this sphere".

Thus, according to this, to calculate the curvature of the gravitational field of a point located at a distance "r_s" from the geometric center that we are considering in our continuous universe, it is only necessary to consider its interaction with the points that are at a radius smaller than "r_s", therefore, the mass "m" to be considered will only be that contained in the sphere of radius "r_s".

In general relativity Birkhoff´s theorem states that any spherically symmetric solution of the vacuum field equations must be statically and asymptotically flat. This means, that the outer solution (that is, the spacetime outside a gravitational, non-rotating, spherical body) must be the Schwarzschild metric.

5. Calculate the curvature of spacetime in a continuous and homogenous medium

We wish to calculate the Gaussian curvature at each point within an extensive, uniform spacetime characterized by a constant mass-energy density, ρ. We consider a sphere of radius R within this spacetime and aim to determine the curvature at a point P on the sphere's surface—a curvature as measured by an observer situated at the sphere's center. As previously discussed, calculating the curvature at a given point in this spacetime requires considering only the interaction between that point and the points located within the sphere. For any point on the surface of the sphere under consideration, this gravitational mass behaves as a point mass—with a magnitude equal to the total mass of the sphere—located at its center. In this scenario, we operate within the framework of the Schwarzschild model, which allows us to employ its equations to calculate the corresponding curvature.

Consequently, we can address the calculation of the Gaussian curvature at points in our spacetime corresponding to an extended universe, at each instant in time, with energy density $\rho(Kg/m^3)$ as a problem solvable via the Schwarzschild model and determine the curvature at interior points using that model. In this model, the static spacetime reduces to a two-dimensional surface, which simplifies the calculation of the Gaussian curvature.

Based on the results before, we have derived an equation relating the Gaussian curvature K_{gauss} of the static spacetime in the Schwarzschild model (J. Droste's solution) to the cosmological parameters of mass M(R) and the universal gravitational constant G. We will use this equation to solve our problem. The equation is as follows:

$$K_{gauss}= -GM(R)/c^2R^3$$

Let us see how this curvature does not depend on the radius of the sphere used to calculate it, but rather on the energy density of the medium.

$$M(R) =(4\pi R^3/3)\ \rho$$

$$K_{gauss}= -GM(R)/c^2R^3 = (4\pi G/3c^2)\ \rho$$

Thus, the Gaussian curvature of spacetime calculated here is the same throughout the medium, depending solely on its energy density.

6. On the rotational speed of stars in a galaxy

Determination of curvatures

First, we shall calculate the curvature of static spacetime at points within a galaxy. To do this, we consider an energy model that includes the stars—the locations for which we wish to calculate the curvature. Let P be a point situated at a distance r from the center of the galaxy. Based on the considerations in sections 4, and 5 regarding the calculation of curvatures, we shall assume an extensive, continuous, and homogeneous medium surrounding the point where a star is located. We also posit an observer situated at the center of the galaxy; our goal is to calculate the curvature at point P with respect to this observer. As established in section 5, the Gaussian curvature is given by:

$$K_{gauss} = \frac{G\,M(r)}{c^2 r^3}$$

We define M(r) as the mass of the stars located at distances less than *r*, calculated using the Sérsic luminosity profile.

To calculate the curvature at other points—relative to the same observer located at the center of the galaxy—we follow the same procedure. At each point, we use the corresponding values of W(r) and r. This yields a value for the curvature at each point, based on the baryonic mass distribution as determined by the Sérsic luminosity profile.

Determination of accelerations

Once the field of curvatures has been determined, we will calculate the field of accelerations produced by those curvatures.

The star's trajectory within the Droste spacetime allows for the calculation of the acceleration it has undergone at any given instant—an acceleration that arises from the curvature of spacetime at each point. Within this inertial framework of spacetime, such acceleration stems from the change in the direction of the star's velocity vector along its path in the spacetime; this centripetal acceleration, within this inertial frame, can be calculated as follows:

$a_{stF} = c^2/R_r$

c is the spacetime speed

R_r it is the radius of curvature of the star's trajectory at any given moment in spacetime.

This radius corresponds to the reciprocal of the curvature of the star's trajectory at that point. Therefore, it can be estimated using the value of the Gaussian curvature at that point (Gaussian curvature is the product of the maximum and minimum curvatures at that point).

$a_{stF} = c^2/R_r$

$(\frac{1}{R_r})^2 \cong K_{gauss}$

$a_{stF} = (\frac{c^2}{R_r})^2 \cong c^4 K_{gauss} = \frac{c^2 GM(r)}{r^3}$

This relativistic acceleration is reflected in the same way in any inertial frame, and therefore also in the comoving inertial frame of our universe.

The equivalence principle allows us to write this equation, which relates the accelerations in spacetime a_{stF} to those in the comoving frame a_{cF} of our universe:

$a^2 = (\frac{V^2}{r})^2{}_{cF} = (\frac{c^2}{R_r})^2{}_{stF}$

cF = conmovil frame (inertial frame)

stF= spacetime frame (inertial frame)

This relativistic acceleration a, it is the cause, as we shall now demonstrate, of the flat rotational speed of stars within the galaxy.

Determination of speeds

$a^2 = \frac{V^4}{r^2} = \frac{c^2 GM(r)}{r^3}$

$M(r) = \frac{V^4 r}{c^2 G}$

As stated at the beginning of this report regarding the relationship between baryonic mass and luminosity for spiral galaxies and $r < r_e$ we can write:

$M(r) \sim L(r) \cong I_e r$

$M(r) = \frac{V^4 r}{c^2 G} \sim I_e r$

$$\mathbf{V^4 \sim I_e\, c^2\, G}$$

Rotation speed of the stars in spiral galaxies for stars at distances from the center of the galaxy less than the effective radius At no time have we used dark matter in our deduction

Thus, we see that the rotation speed of a star does not depend on its position within the galaxy but on factors that are constant in the galaxy; In other words, the rotation speed of stars within the same galaxy is the same for stars that satisfy these distance relations.

The Tully-Fisher relation

$V^4 \sim I_e \sim$ Luminosity$\sim M_{baryonic}$

The equation we have obtained allows us to determine the Tully-Fisher relation. Until now, the Tully-Fisher relation (Tully & Fisher, 1977) had been considered an empirical relationship for spiral galaxies that links luminosity—or baryonic mass—to the fourth power of the stars' rotational velocity; the equation we have derived, $V^4 \sim I_e \sim$ Luminosity $\sim$ M, serves as a demonstration of this relation.

7. Conclusions

Through a novel approach, we have demonstrated the Tully-Fisher relation—a finding that aligns with experimental data. We arrived at this result without invoking dark matter; baryonic matter alone proved sufficient, as demonstrated through a mathematical study of Sérsic profiles and the correct application of the theory of general relativity. We modeled the galaxy as a continuous, extended distribution of energy, with a baryonic mass-energy distribution defined via the study of the Sérsic profile. Within this framework, we calculated the curvature of spacetime at every point, from which we determined the accelerations and velocities of the stars. Our results show that general relativity allows for the identification of a region within the galaxy—specifically, the area inside the effective radius—where stars rotate at the same speed. Consequently, the rotational velocity of these stars is uniform.

The same equation that enables us to determine the flat nature of rotation curves in that region also allows us to derive the Tully-Fisher relation for spiral galaxies. This relation, which was previously purely empirical, now gains a theoretical foundation rooted in the theory of general relativity. In short, by applying the theory of relativity in a novel way, we have succeeded in demonstrating the Tully-Fisher relation. We achieved this using the theory of general relativity, adapted to our galactic model; this theory differs significantly from Newtonian gravitation within the galactic environment and successfully explains certain phenomena that Newtonian gravity could not account for. It is clear that at no point is it necessary to assume the non-existence of dark matter; furthermore, the derivation of the Tully-Fisher relation would remain identical if one were to assume a constant proportionality between dark matter and baryonic matter in the spiral galaxies that are the subject of this study.

8. References

[1] Babcock, H, 1939, "The rotation of the Andromeda Nebula" , ADS Scan Explorer Lick Observatory bulletin no. 498

[2] Melia, Fulvio, 2022, Initial energy of a spatially flat universe: A hint of its possible origin - Astrophysics Data System Astronomische Nachrichten, Vol 343, (3) (2022).

[3] Salmon Iza, F, 2024 A Simple Equation to Calculate the Gaussian Curvature of Space-Time According to the Schwarzschild Model Open Access Journal of Physics. Vol 6 Issue 1. December 2024

[4] Schwarzschild K, On the Gravitational Field of a Mass Point According to Einstein's Theory - Astrophysics Data System 1916. arXiv:physics/9905030v1 [physics.hist-ph] 1999-06-12 (1916)

[5] Sérsic, J. L 1963 "Influence of atmospheric and instrumental scattering on the distribution of brightness in a galaxy". Bulletin of the Argentine Association of Astronomy. **6**: 41–43.

[6] B., Tully, R.; R., Fisher, J. (1977). A new method of determining distances to galaxies. - ADS Astronomy and Astrophysics. **54**: 661. 1977

[7] Vera C. Rubin,1'2 W. Kent Ford, Jr.,1 and Norbert Thonnard, 1980, Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 (R=4kpc) to UGC 2885 (R=122kpc). - Astrophysics Data System 1980, The Astrophysical Journal, 238:471-487, June 1, (1980)

[8] Weinberg Stephen, 1972, Gravitation and cosmology; principles and applications of the general theory of relativity. John Wiley and Sons. Inc New York, London, Sidney, Toronto. 1972.

[9] Wikipedia, 1 Sérsic profile - Wikipedia, (2026)

[10] Wikipedia, 2 Differential geometry of surfaces - Wikipedia, the free encyclopedia (2026)