FRACTALS IN SPECIAL THEORY OF RELATIVITY

U.N.Hladki*

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Abstract

This article employs fractal functions to derive a general equation for the relativistic motion of a particle within an inertial reference frame. In general, relativistic expressions for the law of motion, momentum, energy, and force are described by fractal functions. The general equation of motion explicitly incorporates a mathematical description of relativistic time dilation and the length contraction of a moving body (or particle). When a body (particle) transitions from one equilibrium state ($u_1 = const$) to another ($u_2 = const$), these relativistic effects arise during the period of accelerated motion of the body. Furthermore, the variation in distances and the rate at which coordinate clocks progress — observed when transitioning to a different reference frame — arise from the relative differences between the standards of length and time employed in those respective frames. It should be noted that, in each reference frame, these standards are established according to identical rules and are based on the same physical phenomena. The article also demonstrates that forces of interaction between two moving charges, which are simultaneous in a reference frame S, become non-simultaneous in a reference frame S'. Using this fact, original transformations of forces were performed for cases of both uniform and accelerated motion of charges. In the latter case, fractal functions are also utilized to calculate retarded distances within the Liénard–Wiechert electromagnetic field formulation. The application of fractals to special relativity, along with the resulting findings, is entirely original.

1 Introduction

Let the law of motion of a particle in an inertial reference frame (IRF) S have the form $\mathbf{r}(t)$. A standard problem involving Lorentz transformations is to determine the equation of motion of this

*email: vgladkiy58@gmail.com

particle $\mathbf{r}'(t')$ in a reference frame S' moving with a velocity $\mathbf{v}$. To do this, one must substitute the inverse time transformation $t(t') = f_2[\mathbf{r}'(t'), \mathbf{v}, t']$ into the direct coordinate transformation $\mathbf{r}'(t) = f_1[\mathbf{r}(t), \mathbf{v}, t]$. In the general case, this results in a transcendental equation for the unknown function $\mathbf{r}'(t')$

$$\mathbf{r}'(t') = f_1\left\{\mathbf{r}\left(f_2\left[\mathbf{r}'(t'), \mathbf{v}, t'\right]\right), \mathbf{v}, f_2\left[\mathbf{r}'(t'), \mathbf{v}, t'\right]\right\} . \tag{1.1}$$

In the special case of uniform motion of the particle $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{u}_0 t$, equation (1.1) will be algebraic, and from it one can easily obtain an analytical expression for the function $\mathbf{r}'(t')$. In the general case, the solution to (1.1) is found using numerical methods. That is, we obtain a graph of the function $\mathbf{r}'(t')$. An analytical expression for the general law of particle motion in an IRF has not been obtained. In the article, this problem is solved using fractal functions. This solution to the problem is completely original and has not been used by anyone before.

The situation is similar for the transformation formulas for momentum, energy, and force. To obtain an analytical expression for momentum, energy and force in the frame S' , it is also necessary to use fractal functions.

Let us consider the application of the force transformation formula to the interaction between two particles. When moving to the reference frame S', a difference appears in the moments of time of two events: the action of particle 1 on particle 2 and the action of particle 2 on particle 1. The moment of time t of interaction of two particles in the frame S in the frame S' corresponds to two moments of time t'_1 and t'_2 in accordance with the Lorentz transformations. Also, the distance r_{12} in the frame S corresponds to the distance r'_{11} in the frame S' where the endpoints are defined at different times. Such a distance does not physically exist. An analogy can be drawn here to complex numbers, which are mathematically real yet lack physical existence. It can be assumed that the force exerted by particle 1 on particle 2 in frame S is transformed into a force in frame S' existing at time t'_2, when this event is Lorentz-invariant for particle 2, but not Lorentz-invariant for particle 1. The distance between particles r'_{12} in this case physically exists and pertains to one moment in time t'_2. A similar statement is also true for the transformation of the force of action of particle 1 on particle 2. In this article, calculations are performed confirming this assumption.

Currently, examples given in the literature of finding the forces of interaction of charges in the frame S' based on their forces in the frame S have one common property. A stationary charge q is selected at the origin at time $t = 0$ and the force with which it acts on another charge in a moving frame of reference is found. In such derivations, the time variable is entirely absent. If you try to introduce time into these transformations (for example, to find the interaction with a changing distance),

then without taking into account the desynchronization of events, it is impossible to obtain the correct expression. In this article, the transformation of forces is carried out taking into account the desynchronization of the interaction forces in the frame S' and for any moment of time t from the frame S we obtain the correct expression for the electromagnetic forces in the frame S' . There are no examples of the transformation of interaction forces for the case of accelerated motion of charges in modern scientific literature.

The following designations are used in this article. The IRF S' moves relative to the IRF S with a constant velocity $\mathbf{v}$. In the reference frame S , the initial position of the material particle is $\mathbf{r}_0$, and the initial velocity is $\mathbf{u}_0$. The initially-own reference frame S^* moves relative to the reference frame S with a constant velocity $\mathbf{u}_0$. In this article, the relativistic system of units is used when the speed of light is assumed to be equal to one. We apply Lorentz transformations in vector form.

$$t'(t) = \gamma_v[t - \mathbf{v}\cdot\mathbf{r}(t)] \quad , \quad \text{where} \quad \gamma_v = \frac{1}{\sqrt{1-v^2}} \ , \tag{1.2}$$

$$\mathbf{r}'(t) = \mathbf{r}(t) - \gamma_v\mathbf{v}t + \Gamma_v\mathbf{v}[\mathbf{v}\cdot\mathbf{r}(t)] \quad , \quad \text{where} \quad \Gamma_v = \frac{\gamma_v - 1}{v^2} = \frac{\gamma_v^2}{\gamma_v + 1} \tag{1.3}$$

Fractal functions are written in the form $\underline{\varphi} = \underline{f + g(\,)}$, where $(\,)$ – empty parentheses indicate the place of cyclic substitution of the entire underlined expression. The argument of the function g or the multiplier for g is written in parentheses . The fractal function, expanded into 3 steps, has the form $\underline{\varphi} = f + g(f + g(f + g(\underline{f + g(\,)})))$. Underlining the function $\underline{\varphi}$ or any other function $\underline{F}$ with the argument $\underline{\varphi}$ indicates a fractal function. Underscores may not be used if it is clear from the mathematical expression. The simplest examples of fractal functions are the following $\varphi_1 = \underline{a + b\cdot(\,)}$ and $\varphi_2 = \underline{a + b\cdot(\,) + c\cdot(\,)}$. Their amounts are equal

$$\varphi_1 = \frac{a}{1-b} \quad , \qquad |b| < 1 \quad , \quad \varphi_2 = \frac{a}{1-b-c} \quad , \quad \text{if} \quad |b+c| < 1$$

Fractal functions are, in essence, similar to trigonometric functions. However, while a trigonometric function is represented as an infinite series of terms summed sequentially, a fractal function consists of an infinite number of cycles in which the function is nested within itself. When calculating the value of a trigonometric function to a specified degree of precision, one sums a certain number of terms from the infinite series; similarly, when determining the value of a fractal function to a given level of precision, one uses the function with a specific number of self-nesting cycles.

2 Relativistic uniform motion

Let the equation of motion of a particle in the reference frame S be equal to $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t$. Then law of motion of a particle in the reference frame S'

$$\mathbf{r}' = \mathbf{r}_0 + \frac{\gamma_v \mathbf{u}_0 - \Gamma_v \mathbf{v}}{\gamma_v(1 - \mathbf{v}\cdot\mathbf{u}_0)}(\mathbf{v}\cdot\mathbf{r}_0) + \frac{\mathbf{u}_0 - \gamma_v \mathbf{v} + \Gamma_v \mathbf{v}(\mathbf{v}\cdot\mathbf{u}_0)}{\gamma_v(1 - \mathbf{v}\cdot\mathbf{u}_0)} t' \tag{2.1}$$

A solution to an equation is verified by substituting in (2.1) the direct time transformation (1.2). As a result, we obtain the equation of direct coordinate transformation (1.3).

Let's introduce the notation

$$\mathbf{r}'_0 = \mathbf{r}_0 + \frac{\gamma_v \mathbf{u}_0 - \Gamma_v \mathbf{v}}{\gamma_v(1 - \mathbf{v}\cdot\mathbf{u}_0)}(\mathbf{v}\cdot\mathbf{r}_0) \quad , \quad \mathbf{u}'_0 = \frac{\mathbf{u}_0 - \gamma_v \mathbf{v} + \Gamma_v \mathbf{v}(\mathbf{v}\cdot\mathbf{u}_0)}{\gamma_v(1 - \mathbf{v}\cdot\mathbf{u}_0)}$$

then the equation of motion of a particle with values from the reference frame S' takes the form $\mathbf{r}' = \mathbf{r}'_0 + \mathbf{u}'_0 t'$.

We introduce the following definitions. Equation of motion (2.1) is called *the transitional form of the law of motion of a particle* in the reference frame S'. *The standard view of the law of motion of a particle* should include the initial spatial location and initial velocity from the reference frame that describes the motion of the particle.

3 The general case of relativistic motion

3.1 The equation of relativistic motion without initial velocity

At the initial moment of time, the particle is located in its own reference frame $\mathbf{u}_0 = 0$. Velocity and acceleration of a particle in the reference frame S

$$\mathbf{u}_f = \frac{\mathbf{u}_w}{\sqrt{1 + u_w^2}} \quad , \quad \mathbf{a}_f = \frac{\dot{\mathbf{u}}_w}{(1 + u_w^2)^{3/2}} \quad , \quad \text{where} \quad \mathbf{u}_w = \frac{1}{M}\int \mathbf{F}\, dt$$

The equation of motion of a particle in the reference frame S

$$\mathbf{r} = \mathbf{r}_0 + \int \mathbf{u}_f\, dt = \mathbf{r}_0 + \mathbf{r}_f(t) \quad , \quad \text{where} \quad \mathbf{r}_0 = const \quad , \quad \mathbf{r}_f = r_f(t)\,\mathbf{n} \tag{3.1}$$

Let us find the law of motion of a particle in frame S', which is moving at a velocity $\mathbf{v}$ relative to frame S. From the direct time transformation $t'(t) = \gamma_v\,[t - \mathbf{v}\mathbf{r}_0 - \mathbf{v}\,\mathbf{r}_f(t)]$, we obtain the inverse

function for time $t(t') = \underline{t'/\gamma_v + \mathbf{v}\mathbf{r}_0 + \mathbf{v}\,\mathbf{r}_f(\ )}$, which is a fractal. Since the function $\mathbf{r}_f(t)$ is, by definition, single valued, smooth, and finite, a fractal expression for time t exists. By substituting the fractal into the direct coordinate transformation, we find the law of motion of the particle in frame S'.

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \left[\mathbf{n} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{n})\right] R_f\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right) \tag{3.2}$$

where $\underline{\varphi'} = \underline{\mathbf{v}\cdot\mathbf{r}_0 + \mathbf{v}\cdot\mathbf{R}_f\left(\frac{t'}{\gamma_v} + (\,)\right)}$, $\mathbf{R}_f\left(\frac{t'}{\gamma_v} + (\,)\right)$ – a function $\mathbf{r}_f(t)$ with new argument.

The solution (3.2) is verified by substituting a direct time transformation (1.2) into it. As a result, we get a direct coordinate transformation (1.3).

Similarly to the fractal function $\underline{\mathbf{R}}_f$, we introduce the fractal quantities $\underline{\mathbf{U}}_w$, $\underline{\mathbf{U}}_f$, obtained from the velocities $\mathbf{u}_w$, $\mathbf{u}_f$ by replacing the argument. Velocity of a particle in the reference frame S'

$$\mathbf{u}' = \frac{\gamma_v \underline{\mathbf{U}}_f - \Gamma_v \mathbf{v}\left(\mathbf{v}\cdot\underline{\mathbf{U}}_f\right)}{\gamma_v^2\left(1 - \mathbf{v}\cdot\underline{\mathbf{U}}_f\right)} - \mathbf{v} \quad , \quad (u')^2 = 1 - \frac{(1-v^2)\left(1 - \underline{U}_f^2\right)}{\left(1 - \mathbf{v}\cdot\underline{\mathbf{U}}_f\right)} < 1$$

If $\left[\mathbf{v}\times\underline{\mathbf{R}}_f\right] = 0$, then expression (3.2) takes the form

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \frac{1}{\gamma_v}\mathbf{R}_f\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right) \tag{3.3}$$

and when $(\mathbf{v}\cdot\underline{\mathbf{R}}_f) = 0$, then the next

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \mathbf{R}_f\left(\frac{t'}{\gamma_v} + \mathbf{v}\cdot\mathbf{r}_0\right) \tag{3.4}$$

We can switch from the vector form of writing the equation of motion (3.2) to the coordinate form. The simplest expressions are obtained when switching to a rectilinear oblique coordinate system with unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The cosines of the angles between the unit vectors are equal

$$\cos(\mathbf{e}_1, \mathbf{e}_2) = -\gamma_{v_x}\gamma_{v_y} v_x v_y \quad , \quad \cos(\mathbf{e}_1, \mathbf{e}_3) = -\gamma_{v_x}\gamma_{v_z} v_x v_z \quad , \quad \cos(\mathbf{e}_2, \mathbf{e}_3) = -\gamma_{v_y}\gamma_{v_z} v_y v_z$$

$$\text{where} \quad \gamma_{v_x} = \frac{1}{\sqrt{1 - v_x^2}} \quad , \quad \gamma_{v_y} = \frac{1}{\sqrt{1 - v_y^2}} \quad , \quad \gamma_{v_z} = \frac{1}{\sqrt{1 - v_z^2}}$$

The matrix M1 defines the transition from the Cartesian coordinates of equation (3.2) to the contravariant coordinates of equation (3.6)

$$M1 = \begin{pmatrix} \frac{\gamma_v}{\gamma_{v_x}} - \frac{\gamma_v^2\left(v_y^2+v_z^2\right)}{\gamma_{v_x}\left(\gamma_v+1\right)} & \frac{\gamma_v^2 v_x v_y}{\gamma_{v_x}\left(\gamma_v+1\right)} & \frac{\gamma_v^2 v_x v_z}{\gamma_{v_x}\left(\gamma_v+1\right)} \\ \frac{\gamma_v^2 v_x v_y}{\gamma_{v_y}\left(\gamma_v+1\right)} & \frac{\gamma_v}{\gamma_{v_y}} - \frac{\gamma_v^2\left(v_x^2+v_z^2\right)}{\gamma_{v_y}\left(\gamma_v+1\right)} & \frac{\gamma_v^2 v_y v_z}{\gamma_{v_y}\left(\gamma_v+1\right)} \\ \frac{\gamma_v^2 v_x v_z}{\gamma_{v_z}\left(\gamma_v+1\right)} & \frac{\gamma_v^2 v_y v_z}{\gamma_{v_z}\left(\gamma_v+1\right)} & \frac{\gamma_v}{\gamma_{v_z}} - \frac{\gamma_v^2\left(v_x^2+v_y^2\right)}{\gamma_{v_z}\left(\gamma_v+1\right)} \end{pmatrix} \tag{3.5}$$

$$\begin{aligned} \mathbf{r}' \;=\; & \left[x_0 - \gamma_v v_x t' + R_{f_x}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right)\right]\frac{\mathbf{e}_1}{\gamma_{v_x}} + \left[y_0 - \gamma_v v_y t' + R_{f_y}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right)\right]\frac{\mathbf{e}_2}{\gamma_{v_y}} + \\ & + \left[z_0 - \gamma_v v_z t' + R_{f_z}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right)\right]\frac{\mathbf{e}_3}{\gamma_{v_z}} \end{aligned} \tag{3.6}$$

The matrix M2 defines the transition from the Cartesian coordinates of equation (3.2) to the covariant coordinates of equation (3.8)

$$M2 = \frac{-1}{\gamma_v+1}\begin{pmatrix} \gamma_{v_x}\gamma_v v_x^2 - \gamma_{v_x}(1+\gamma_v) & \gamma_{v_x}\gamma_v v_x v_y & \gamma_{v_x}\gamma_v v_x v_z \\ \gamma_{v_y}\gamma_v v_x v_y & \gamma_{v_y}\gamma_v v_y^2 - \gamma_{v_y}(1+\gamma_v) & \gamma_{v_y}\gamma_v v_y v_z \\ \gamma_{v_z}\gamma_v v_x v_z & \gamma_{v_z}\gamma_v v_y v_z & \gamma_{v_z}\gamma_v v_z^2 - \gamma_{v_z}(1+\gamma_v) \end{pmatrix} \tag{3.7}$$

$$\det M1 \cdot \det M2 = 1 \quad , \quad M1 \cdot M2^T = E$$

$$\begin{aligned} \mathbf{r}' \;=\; & \gamma_{v_x}\left[x_0 - v_x\mathbf{v}\cdot\mathbf{r}_0 - \frac{v_x}{\gamma_v}t' + R_{f_x}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right) - v_x\mathbf{v}\cdot\mathbf{R}_{f_x}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right)\right]\mathbf{e}_1 + \\ & + \gamma_{v_y}\left[y_0 - v_y\mathbf{v}\cdot\mathbf{r}_0 - \frac{v_y}{\gamma_v}t' + R_{f_y}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right) - v_y\mathbf{v}\cdot\mathbf{R}_{f_y}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right)\right]\mathbf{e}_2 + \\ & + \gamma_{v_z}\left[z_0 - v_z\mathbf{v}\cdot\mathbf{r}_0 - \frac{v_z}{\gamma_v}t' + R_{f_z}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right) - v_z\mathbf{v}\cdot\mathbf{R}_{f_z}\left(\frac{t'}{\gamma_v} + \underline{\varphi'}\right)\right]\mathbf{e}_3 \end{aligned} \tag{3.8}$$

Let the equation of motion (3.1) have the form $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t$. Then the law of motion in the reference frame S' in accordance with formula (3.2) is equal to

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v\mathbf{v}}{\gamma_v+1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \left[\mathbf{u}_0 - \frac{\gamma_v\mathbf{v}}{\gamma_v+1}(\mathbf{v}\cdot\mathbf{u}_0)\right]\left(\underline{\frac{t'}{\gamma_v} + \mathbf{v}\cdot\mathbf{r}_0 + \mathbf{v}\cdot\mathbf{u}_0\cdot(\,)}\right)$$

When summing the fractal, we get the formula (2.1) .

We transform the transition equation of motion (3.2) to the standard form. Since the particle moves

in the reference frame S' with a constant velocity $\mathbf{u}'_0 = -\mathbf{v}$, the second component of the velocity cannot be any. Its value is limited by the magnitude and direction of the velocity $\mathbf{u}'_0$. Therefore, the standard form of the law of motion of a particle cannot have the usual form $\mathbf{r}' = \mathbf{r}'_0 + \mathbf{u}'_0 t' + \mathbf{r}'_f(t')$. The standard form of the equation of motion of a particle is $\mathbf{r}' = \mathbf{r}'_0 + \mathbf{u}'_0 t' + K(u'_0)\, R'_f(t')\, \mathbf{n}'$. The direction of the vector $\mathbf{R}'_f$ is given by the unit vector $\mathbf{n}'$

$$\mathbf{n}' = \gamma_{v_n}\left[\mathbf{n} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{n})\right] \quad , \quad \text{where} \quad \gamma_{v_n} = \frac{1}{\sqrt{1-(\mathbf{v}\cdot\mathbf{n})^2}} \tag{3.9}$$

Inverse transformation of unit vectors

$$\mathbf{n} = \frac{\gamma_{u'_{\perp n'}}}{\gamma_{u'}}\left[\mathbf{n}' + \Gamma_{u'}(\mathbf{u}'_0\cdot\mathbf{n}')\mathbf{u}'_0\right] \quad , \quad \text{where} \quad \gamma_{u'_{\perp n'}} = \frac{1}{\sqrt{1-(u'_0)^2 + (\mathbf{u}'_0\cdot\mathbf{n}')^2}} \tag{3.10}$$

Taking into account the ratios

$$\underline{r}'_f = \frac{\gamma_{u'_{\perp n'}}}{\gamma_{u'}}\underline{R}_f \ , \ \mathbf{r}'_0 = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) \ , \ \mathbf{v}\cdot\underline{\mathbf{R}}_f = -\gamma_{u'_{\perp n'}}\underline{R}_f \mathbf{u}'_0\cdot\mathbf{n}' \ , \ \mathbf{v}\cdot\mathbf{r}_0 = -\gamma_{u'}\mathbf{u}'_0\cdot\mathbf{r}'_0$$

we obtain the standard form of the relativistic equation of motion of a particle (omit the strokes)

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t + \frac{\gamma_{u_{\perp n}}}{\gamma_u} R_f\left(\frac{t}{\gamma_u} - \underline{\varphi}\right)\mathbf{n} \ , \ \text{where} \ \underline{\varphi} = \underline{\gamma_u \mathbf{u}_0\cdot\mathbf{r}_0 + \gamma_{u_{\perp n}}\mathbf{u}_0\cdot\mathbf{n}\, R_f\left(\frac{t}{\gamma_u} - (\,)\right)} \tag{3.11}$$

where $R_f(t)$ – is a function of motion in the initially-own reference frame S^*.

If $[\mathbf{u}_0 \times \mathbf{n}] = 0$, then expression (3.11) is transformed to the form

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t + \frac{1}{\gamma_u}\mathbf{R}_f\left(\frac{t}{\gamma_u} - \underline{\varphi}\right) \quad , \quad \text{where} \quad \underline{\varphi} = \underline{\gamma_u \mathbf{u}_0\cdot\mathbf{r}_0 + \mathbf{u}_0\cdot\mathbf{R}_f\left(\frac{t}{\gamma_u} - (\,)\right)} \quad , \tag{3.12}$$

and when $(\mathbf{u}_0\cdot\mathbf{n}) = 0$, then

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t + \mathbf{R}_f\left(\frac{t}{\gamma_u} - \gamma_u \mathbf{u}_0\cdot\mathbf{r}_0\right) \tag{3.13}$$

For cases (3.12) - (3.13) , the direction of the vector $\mathbf{R}_f$ coincides with its direction in the initially-own reference frame.

Differentiating (3.11), we find the velocity of the particle

$$\mathbf{u} = \mathbf{u}_0 + \frac{\gamma_{u_{\perp n}}\underline{U}_f \mathbf{n}}{\gamma_u^2\left(1 + \gamma_{u_{\perp n}}\underline{U}_f \mathbf{u}_0\cdot\mathbf{n}\right)} \ , \ u^2 = 1 - \frac{\left(1-u_0^2\right)\left(1-\underline{U}_f^2\right)}{\left(1 + \gamma_{u_{\perp n}}\underline{U}_f \mathbf{u}_0\cdot\mathbf{n}\right)^2} < 1 \quad , \quad |\gamma_{u_{\perp n}}\mathbf{u}_0\cdot\mathbf{n}| \le u_0$$

where functions $\underline{U}_f, \underline{U}_w$ have an argument $\left(\frac{t}{\gamma_u} - \underline{\varphi}\right)$.

3.2 The equation of relativistic motion with initial velocity

We use the symbol $\mathbf{U}_b$ to indicate the initial velocity. The velocity of a particle in the reference frame S

$$\mathbf{u} = \mathbf{u}_f + \mathbf{u}_g \ , \ \mathbf{u}_f = \frac{\mathbf{u}_w}{\sqrt{1 + (\mathbf{U}_b + \mathbf{u}_w)^2}} \ , \ \mathbf{u}_g = \frac{\mathbf{U}_b}{\sqrt{1 + (\mathbf{U}_b + \mathbf{u}_w)^2}} \ , \ \mathbf{n} = \frac{\mathbf{u}_w}{u_w} \ , \ \mathbf{m} = \frac{\mathbf{U}_b}{U_b}$$

The equation of motion of a particle in the reference frame S

$$\mathbf{r} = \mathbf{r}_0 + \int \frac{\mathbf{U}_b \, dt}{\sqrt{1 + (\mathbf{U}_b + \mathbf{u}_w)^2}} + \int \frac{\mathbf{u}_w \, dt}{\sqrt{1 + (\mathbf{U}_b + \mathbf{u}_w)^2}} = \mathbf{r}_0 + \mathbf{r}_g(t) + \mathbf{r}_f(t) \tag{3.14}$$

From the Lorentz transformations, we obtain the equation of motion of a particle in the reference frame S'

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \left[\mathbf{m} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{m})\right] R_g\left(\frac{t'}{\gamma_v} + \underline{\varphi}'\right) + \\ + \left[\mathbf{n} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{n})\right] R_f\left(\frac{t'}{\gamma_v} + \underline{\varphi}'\right) \tag{3.15}$$

$$\text{where} \quad \underline{\varphi}' = \underline{\mathbf{v}\mathbf{r}_0 + \mathbf{v}\cdot\mathbf{R}_g\left(\frac{t'}{\gamma_v} + (\,)\right) + \mathbf{v}\cdot\mathbf{R}_f\left(\frac{t'}{\gamma_v} + (\,)\right)} \quad ,$$

$$\mathbf{R}_g\left(\frac{t'}{\gamma_v} + \underline{\varphi}'\right) - \quad \text{function } \mathbf{r}_g(t) \text{with a new argument.}$$

The vector form of the equation of motion (3.15) can be converted to a coordinate one. The matrix M1 (3.5) provides a transition to contravariant coordinates, and the matrix M2 (3.7) provides a transition to covariant coordinates.

The transitional law of motion of a particle (3.15) is transformed to a standard form and the symbols included in it are written without strokes. The general expression for the equation of motion of a particle in an IRF is

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t + \frac{\gamma_{u_{\perp m}}}{\gamma_u} R_g\left(\frac{t}{\gamma_u} - \underline{\varphi}\right)\mathbf{m} + \frac{\gamma_{u_{\perp n}}}{\gamma_u} R_f\left(\frac{t}{\gamma_u} - \underline{\varphi}\right)\mathbf{n} \quad , \tag{3.16}$$

$$\text{where} \quad \underline{\varphi} = \underline{\gamma_u \mathbf{u}_0\cdot\mathbf{r}_0 + \gamma_{u_{\perp m}} \mathbf{u}_0\cdot\mathbf{m} \, R_g\left(\frac{t}{\gamma_u} - (\,)\right) + \gamma_{u_{\perp n}} \mathbf{u}_0\cdot\mathbf{n} \, R_f\left(\frac{t}{\gamma_u} - (\,)\right)}$$

The direction of the vectors from (3.16) differs from the direction of analogous vectors in the initial reference frame S^* . If n is the direction of a vector in a given reference frame S , and $\mathbf{n}^*$ is the direction in the reference frame S^* , then the coupling equations for unit vectors are (3.9) - (3.10) . The unit vectors $\mathbf{m}$ and $\mathbf{m}^*$ are related by analogous equations.

4 Relativistic uniformly accelerated motion

The differential equation for relativistic motion in a constant electric field is

$$M\frac{d}{dt}\frac{\mathbf{u}_f}{\sqrt{1-u_f^2}} = q\mathbf{E}_0 \quad , \quad \mathbf{E}_0 = const$$

4.1 The solution under the condition of zero initial velocity

The velocity and acceleration of the particle's motion in the reference frame S

$$\mathbf{u}_f = \frac{\mathbf{W}t}{\sqrt{1+W^2t^2}} \quad , \quad \mathbf{a}_f = \frac{\mathbf{W}}{(1+W^2t^2)^{3/2}} \quad , \quad \text{where} \quad \mathbf{W} = \frac{q}{M}\mathbf{E}_0$$

The equation of motion of a particle

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{n}}{W}\left(\sqrt{1+W^2t^2}-1\right) \quad , \quad \mathbf{n} = \frac{\mathbf{W}}{W}$$

Let's switch to the reference frame S'. From formula (3.2) we obtain the transitional form of the law of motion of a particle in the reference frame S'

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v\mathbf{v}}{\gamma_v+1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \frac{1}{W}\left[\mathbf{n} - \frac{\gamma_v\mathbf{v}}{\gamma_v+1}(\mathbf{v}\cdot\mathbf{n})\right]\left[\sqrt{1+W^2\left(\gamma_v^{-1}t'+\underline{\varphi'}\right)^2}-1\right] \tag{4.1}$$

$$\text{where} \quad \underline{\varphi'} = \underline{\mathbf{v}\cdot\mathbf{r}_0 + \frac{\mathbf{v}\cdot\mathbf{n}}{W}\left[\sqrt{1+W^2\left(\gamma_v^{-1}t'+(\,)\right)^2}-1\right]}$$

From formula (3.11) we find the standard form of the law of relativistic uniformly accelerated motion of a particle

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0t + \frac{\gamma_{u_{\perp n}}\mathbf{n}}{\gamma_u W}\left[\sqrt{1+W^2\left(\gamma_u^{-1}t-\underline{\varphi}\right)^2}-1\right] \tag{4.2}$$

$$\text{where} \quad \underline{\varphi} = \underline{\gamma_u\mathbf{u}_0\cdot\mathbf{r}_0 + \gamma_{u_{\perp n}}\frac{\mathbf{u}_0\cdot\mathbf{n}}{W}\left[\sqrt{1+W^2\left(\gamma_u^{-1}t-(\,)\right)}-1\right]}$$

The fractal of the square root is summed up to an expression in elementary functions

$$\underline{\sqrt{1+(at+b+c\cdot(\,)\,)^2}-1}=\frac{1}{1-c^2}\left[bc+act+\sqrt{1-c^2+(at+b-c)^2}-1\right]$$

Let's sum up the fractal in formula (4.1) . The transitional form of the law of uniformly accelerated motion of a particle (4.1) takes the form

$$\mathbf{r}'=\mathbf{r}_0-\frac{\gamma_v\mathbf{v}}{\gamma_v+1}(\mathbf{v}\cdot\mathbf{r}_0)+\gamma_v(\mathbf{v}\cdot\mathbf{r}_0)(\mathbf{v}\cdot\mathbf{n}')\mathbf{n}'-[\mathbf{v}-(\mathbf{v}\cdot\mathbf{n}')\mathbf{n}']\,t'+$$
$$+\frac{\mathbf{n}'}{W}\left[\sqrt{1+\gamma_{v_n}^2(\gamma_v^{-1}Wt'+W\mathbf{v}\cdot\mathbf{r}_0-\mathbf{v}\cdot\mathbf{n})^2}-\gamma_{v_n}\right]\qquad(4.3)$$

where $\mathbf{n}'$ is found by formula (3.9) .

Let's sum up the fractal in the equation of motion (4.2) and get the following expression

$$\mathbf{r}=\mathbf{r}_0+\gamma_u^2\,(\mathbf{u}_0\cdot\mathbf{r}_0)(\mathbf{u}_0\cdot\mathbf{n})\mathbf{n}+[\mathbf{u}_0-(\mathbf{u}_0\cdot\mathbf{n})\mathbf{n}]\,t+$$
$$+\frac{\mathbf{n}}{W}\left[\sqrt{1+\left(\gamma_{u_{\perp n}}^{-1}Wt-\gamma_{u_{\perp n}}^{-1}\gamma_u^2\,W\mathbf{u}_0\cdot\mathbf{r}_0+\gamma_u\mathbf{u}_0\cdot\mathbf{n}\right)^2}-\frac{\gamma_u}{\gamma_{u_{\perp n}}}\right]\qquad(4.4)$$

The initial spatial location of the particle is denoted as $\mathbf{r}_0^*$. It is equal to $\mathbf{r}_0^*=\mathbf{r}_0+\gamma_u^2(\mathbf{u}_0\cdot\mathbf{r}_0)(\mathbf{u}_0\cdot\mathbf{n})\mathbf{n}$. Replace the vector $\mathbf{r}_0$ with the vector $\mathbf{r}_0^*$ in (4.4) and discard the asterisks. Then the standard form of the uniformly accelerated motion equation of particle is

$$\mathbf{r}=\mathbf{r}_0+[\mathbf{u}_0-(\mathbf{u}_0\cdot\mathbf{n})\mathbf{n}]\,t+\frac{\mathbf{n}}{W}\left[\sqrt{1+\left(\gamma_{u_{\perp n}}^{-1}Wt-\gamma_{u_{\perp n}}W\mathbf{u}_0\cdot\mathbf{r}_0+\gamma_u\mathbf{u}_0\cdot\mathbf{n}\right)^2}-\frac{\gamma_u}{\gamma_{u_{\perp n}}}\right]\qquad(4.5)$$

Velocity of a particle

$$\mathbf{u}=\mathbf{u}_0-(\mathbf{u}_0\cdot\mathbf{n})\mathbf{n}+\frac{\left(\gamma_{u_{\perp n}}^{-1}Wt-\gamma_{u_{\perp n}}W\mathbf{u}_0\cdot\mathbf{r}_0+\gamma_u\mathbf{u}_0\cdot\mathbf{n}\right)\mathbf{n}}{\gamma_{u_{\perp n}}^2\sqrt{1+\left(\gamma_{u_{\perp n}}^{-1}Wt-\gamma_{u_{\perp n}}W\mathbf{u}_0\cdot\mathbf{r}_0+\gamma_u\mathbf{u}_0\cdot\mathbf{n}\right)^2}}$$

$$u^2=1-\frac{1}{\gamma_{u_{\perp n}}^2\left[1+\left(\gamma_{u_{\perp n}}^{-1}Wt-\gamma_{u_{\perp n}}W\mathbf{u}_0\cdot\mathbf{r}_0+\gamma_u\mathbf{u}_0\cdot\mathbf{n}\right)^2\right]}<1$$

Under the condition $[\mathbf{u}_0\times\mathbf{n}]=0$, expression (4.5) takes the form

$$\mathbf{r}=\mathbf{r}_0+\frac{\mathbf{n}}{W}\left[\sqrt{1+\left(Wt-W\mathbf{u}_0\cdot\mathbf{r}_0+\gamma_u\mathbf{u}_0\cdot\mathbf{n}\right)^2}-\gamma_u\right]$$

and when $(\mathbf{u}_0 \cdot \mathbf{n}) = 0$ it takes the following form

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t + \frac{\mathbf{n}}{W}\left[\sqrt{1 + W^2\left(\gamma_u^{-1} t - \gamma_u \mathbf{u}_0 \cdot \mathbf{r}_0\right)^2} - 1\right]$$

For uniformly accelerated motion , the functional dependence on time is reduced by a factor of $1/\gamma_{u_{\perp n}}$. The time dilation depends on the component of the velocity perpendicular to the direction of accelerated motion. The perpendicular component of the velocity remains constant in time, while the longitudinal component changes. In equation (4.5), the longitudinal component of the initial velocity $(\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}$ is included in the function of uniformly accelerated motion. This is easy to show if we take the limit from this function at $W \to 0$

$$\lim_{W \to 0} \frac{\mathbf{n}}{W}\left[\sqrt{1 + \left(\gamma_{u_{\perp n}}^{-1} W t - \gamma_{u_{\perp n}} W \mathbf{u}_0 \cdot \mathbf{r}_0 + \gamma_u \mathbf{u}_0 \cdot \mathbf{n}\right)^2} - \frac{\gamma_u}{\gamma_{u_{\perp n}}}\right] = (\mathbf{v} \cdot \mathbf{n})\mathbf{n} t - \gamma_{u_{\perp n}}^2 (\mathbf{u}_0 \cdot \mathbf{r}_0)\mathbf{n}$$

or if we switch to the reference frame S^* , which moves relative to the given frame at velocity $\mathbf{u}_0$, and not at the velocity $[\mathbf{u}_0 - (\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}]$, as it may seem from equation (4.5). The equation of motion of a particle in the reference frame S^* has the form

$$\mathbf{r}' = \mathbf{r}_0 + \gamma_{u_{\perp n}}^2 (\mathbf{u}_0 \cdot \mathbf{r}_0)\left[\frac{\mathbf{u}_0}{\gamma_u + 1} - (\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}\right] + \frac{\gamma_{u_{\perp n}}}{\gamma_u W}\left[\mathbf{n} + \Gamma_u \mathbf{u}_0 (\mathbf{u}_0 \cdot \mathbf{n})\right]\left[\sqrt{1 + (Wt')^2} - 1\right]$$

4.2 The solution under the condition of non-zero initial velocity

The initial velocity is indicated by the symbol $\mathbf{U}_b$. Velocity and the law of motion of a particle in the reference frame S

$$\mathbf{u} = \frac{\mathbf{U}_b + \mathbf{W} t}{\sqrt{1 + (\mathbf{U}_b + \mathbf{W} t)^2}}$$

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{n}}{\gamma_{b_n} W}\left[\sqrt{1 + \left(\gamma_{b_n} W t + \gamma_{b_n} \mathbf{U}_b \cdot \mathbf{n}\right)^2} - \gamma_{b_n} \gamma_b\right] + \\ + \frac{1}{W}\left[\mathbf{U}_b - (\mathbf{U}_b \cdot \mathbf{n})\,\mathbf{n}\right] \mathrm{ash}(\gamma_{b_n} W t + \gamma_{b_n} \mathbf{U}_b \cdot \mathbf{n}) \quad (4.6)$$

$$\text{where} \quad n = \frac{\mathbf{W}}{W} \quad , \quad \gamma_{b_n} = \frac{1}{\sqrt{1 + U_b^2 - (\mathbf{U}_b \mathbf{n})^2}} \quad , \quad \gamma_b = \sqrt{1 + U_b^2}$$

Let's switch to the reference frame S'. We obtain the equation of motion of a particle in the reference frame S'

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \frac{1}{W}\left[\mathbf{U}_{b_{\perp n}} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{U}_{b_{\perp n}})\right] \mathrm{ash}\left(\frac{\gamma_{b_n}}{\gamma_v} W t' + \gamma_{b_n}\mathbf{U}_b\cdot\mathbf{n} + \underline{\varphi'}\right) + \frac{1}{\gamma_{b_n} W}\left[\mathbf{n} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{n})\right]\left[\sqrt{1 + \left(\gamma_v^{-1}\gamma_{b_n} W t' + \gamma_{b_n}\mathbf{U}_b\cdot\mathbf{n} + \underline{\varphi'}\right)^2} - \gamma_b\gamma_{b_n}\right] \quad (4.7)$$

$$\text{where} \quad \underline{\varphi'} = \underline{\gamma_{b_n} W \mathbf{v}\cdot\mathbf{r}_0 + \mathbf{v}\cdot\mathbf{n}\left[\sqrt{1 + \left(\gamma_v^{-1}\gamma_{b_n} W t' + \gamma_{b_n}\mathbf{U}_b\cdot\mathbf{n} + (\,)\right)^2} - \gamma_b\gamma_{b_n}\right] + \gamma_{b_n}\mathbf{v}\cdot\mathbf{U}_{b_{\perp n}}\mathrm{ash}\left(\gamma_v^{-1}\gamma_{b_n} W t' + \gamma_{b_n}\mathbf{U}_b\cdot\mathbf{n} + (\,)\right)} \quad , \quad \mathbf{U}_{b_{\perp n}} = \mathbf{U}_b - (\mathbf{U}_b\cdot\mathbf{n})\mathbf{n}$$

Velocity of a particle

$$\mathbf{u}' = \frac{\gamma_v \underline{\mathbf{U}} - \Gamma_v \mathbf{v}(\mathbf{v}\cdot\underline{\mathbf{U}})}{\gamma_v^2\,(1 - \mathbf{v}\cdot\underline{\mathbf{U}})} - \mathbf{v} \quad , \quad \text{where} \quad \underline{\mathbf{U}} = \frac{\gamma_{b_n}\mathbf{U}_b + \left(\gamma_v^{-1}\gamma_{b_n} W t' + \underline{\varphi'}\right)\mathbf{n}}{\left[1 + \left(\gamma_v^{-1}\gamma_{b_n} W t' + \gamma_{b_n}\mathbf{U}_b\cdot\mathbf{n} + \underline{\varphi'}\right)^2\right]^{1/2}}$$

Provided that $[\mathbf{U}_b \times \mathbf{n}] = 0$ the equation of motion in the reference frame S (4.6) is transformed to the form

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{n}}{W}\left[\sqrt{1 + (Wt + U_b)^2} - \gamma_b\right]$$

and in the reference frame S', the equation of motion (4.7) has the form

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \frac{1}{W}\left[\mathbf{n} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{n})\right]\left[\sqrt{1 + \left(\gamma_v^{-1} W t' + U_b + \underline{\varphi'}\right)^2} - \gamma_b\right]$$

$$\text{where} \quad \underline{\varphi'} = \underline{W \mathbf{v}\cdot\mathbf{r}_0 + \mathbf{v}\cdot\mathbf{n}\left[\sqrt{1 + \left(\gamma_v^{-1} W t' + U_b + (\,)\right)^2} - \gamma_b\right]}$$

Under the condition $(\mathbf{U}_b\cdot\mathbf{n}) = 0$, the equation of motion in the reference frame S (4.6) is transformed to the form

$$\mathbf{r} = \mathbf{r}_0 + \frac{\gamma_b \mathbf{n}}{W}\left[\sqrt{1 + (\gamma_b^{-1} W t)^2} - 1\right]$$

and in the reference frame S', the equation of motion (4.7) is

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' + \frac{\gamma_b}{W}\left[\mathbf{n} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{n})\right]\left[\sqrt{1 + \left(\gamma_v^{-1}\gamma_b^{-1}Wt' + \underline{\varphi'}\right)^2} - 1\right] + \\ + \frac{1}{W}\left[\mathbf{U}_b - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{U}_b)\right]\mathrm{ash}\left(\gamma_v^{-1}\gamma_b^{-1}Wt' + \underline{\varphi'}\right) \quad ,$$

$$\text{where} \quad \underline{\varphi'} = \underline{\frac{W}{\gamma_b}\mathbf{v}\cdot\mathbf{r}_0 + \mathbf{v}\cdot\mathbf{n}\left[\sqrt{1 + \left(\gamma_v^{-1}\gamma_b^{-1}Wt' + (\,)\right)^2} - 1\right] + \frac{\mathbf{v}\cdot\mathbf{U}_b}{\gamma_b}\mathrm{ash}\left(\gamma_v^{-1}\gamma_b^{-1}Wt' + (\,)\right)}$$

The transitional form of the law of motion of the particle (4.7) is transformed into a standard one, the values of which are recorded without strokes.

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t + \frac{\gamma_{u_{\perp n}}\mathbf{n}}{\gamma_{b_{n^*}}\gamma_u W}\left[\sqrt{1 + \left(\gamma_u^{-1}\gamma_{b_{n^*}}Wt + \gamma_{b_{n^*}}\mathbf{U}_b\cdot\mathbf{n}^* - \underline{\varphi}\right)^2} - \gamma_b\gamma_{b_{n^*}}\right] + \\ + \frac{1}{\gamma_u W}\left[\gamma_{u_{\perp m}}U_b\mathbf{m} - \gamma_{u_{\perp n}}(\mathbf{U}_b\cdot\mathbf{n}^*)\mathbf{n}\right]\mathrm{ash}\left(\gamma_u^{-1}\gamma_{b_{n^*}}Wt + \gamma_{b_{n^*}}\mathbf{U}_b\cdot\mathbf{n}^* - \underline{\varphi}\right) \qquad (4.8)$$

$$\text{where} \quad \underline{\varphi} = \underline{\gamma_{b_{n^*}}\gamma_u W\mathbf{u}_0\cdot\mathbf{r}_0 + \gamma_{u_{\perp n}}\mathbf{u}_0\cdot\mathbf{n}\left[\sqrt{1 + \left(\gamma_u^{-1}\gamma_{b_{n^*}}Wt + \gamma_{b_{n^*}}\mathbf{U}_b\cdot\mathbf{n}^* - (\,)\right)^2} - \gamma_b\gamma_{b_{n^*}}\right] +}$$

$$\underline{+\gamma_{b_{n^*}}\left[\gamma_{u_{\perp m}}U_b\mathbf{u}_0\cdot\mathbf{m} - \gamma_{u_{\perp n}}(\mathbf{U}_b\cdot\mathbf{n}^*)\mathbf{n}\right]\mathrm{ash}\left(\gamma_u^{-1}\gamma_{b_{n^*}}Wt + \gamma_{b_{n^*}}\mathbf{U}_b\cdot\mathbf{n}^* - (\,)\right)}$$

$$\gamma_{b_{n^*}} = \frac{1}{\sqrt{1 + U_b^2 - (\mathbf{U}_b\cdot\mathbf{n}^*)^2}} \;\; ; \;\; \mathbf{U}_b \;, \;\; \mathbf{n}^* - \text{values from the initialy-own reference frame } S^*.$$

5 Relativistic periodic motion

The differential equation for relativistic rotational (elliptic) motion in its own reference frame $\mathbf{u}_0$ is

$$M\frac{d}{dt}\frac{\mathbf{u}_f}{\sqrt{1 - u_f^2}} = q\mathbf{E}_1\cos\omega t + q\mathbf{E}_2\sin\omega t \quad , \quad (\mathbf{E}_1\cdot\mathbf{E}_2) = 0 \quad , \quad |\mathbf{E}_1| \neq |\mathbf{E}_2|$$

Velocity of a particle in the reference frame S

$$\mathbf{u}_f = \frac{\mathbf{U}_1\sin\omega t - \mathbf{U}_2\cos\omega t}{\sqrt{1 + (U_1\sin\omega t)^2 + (U_2\cos\omega t)^2}} \quad , \quad \text{where} \quad \mathbf{U}_1 = \frac{q}{M\omega}\mathbf{E}_1 \quad , \quad \mathbf{U}_2 = \frac{q}{M\omega}\mathbf{E}_2$$

The equation of motion is

$$\mathbf{r} = \mathbf{r}_0 - \frac{\mathbf{U}_1}{\omega U_{12}} \arcsin\left(\frac{U_{12}}{\sqrt{1+U_1^2}} \cos\omega t\right) - \frac{\mathbf{U}_2}{\omega U_{21}} \arcsin\left(\frac{U_{21}}{\sqrt{1+U_2^1}} \sin\omega t\right)$$

$$\text{where} \quad U_{12} = \sqrt{U_1^2 - U_2^2} \quad , \quad U_{21} = \sqrt{U_2^2 - U_1^2} \quad , \quad \mathbf{n} = \frac{\mathbf{U}_1}{U_1} \quad , \quad \mathbf{m} = \frac{\mathbf{U}_2}{U_2}$$

From formula (3.2) we obtain the transitional form of the equation of relativistic rotation and uniform displacement

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' - \frac{U_1}{\omega U_{12}} \left[\mathbf{n} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{n})\right] \arcsin\left[\frac{U_{12}}{\sqrt{1+U_1^2}} \cos\left(\frac{\omega t'}{\gamma_v} + \underline{\varphi}'\right)\right] -$$
$$- \frac{U_2}{\omega U_{21}} \left[\mathbf{m} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{m})\right] \arcsin\left[\frac{U_{21}}{\sqrt{1+U_2^2}} \sin\left(\frac{\omega t'}{\gamma_v} + \underline{\varphi}'\right)\right] \tag{5.1}$$

$$\text{where} \quad \underline{\varphi}' = \omega \mathbf{v}\cdot\mathbf{r}_0 - \frac{U_1}{\omega U_{12}} \arcsin\left[\frac{U_{12}}{\sqrt{1+U_1^2}} \cos\left(\frac{\omega t'}{\gamma_v} + (\,)\right)\right] -$$
$$- \frac{U_2}{\omega U_{21}} \arcsin\left[\frac{U_{21}}{\sqrt{1+U_2^2}} \sin\left(\frac{\omega t'}{\gamma_v} + (\,)\right)\right]$$

The standard form of the equation of relativistic rotation and displacement is found by formula (3.11)

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t - \frac{\gamma_{u_{\perp n}} U_1 \mathbf{n}}{\gamma_u \omega U_{12}} \arcsin\left[\frac{U_{12}}{\sqrt{1+U_1^2}} \cos\left(\frac{\omega t}{\gamma_u} - \underline{\varphi}\right)\right] -$$
$$- \frac{\gamma_{u_{\perp m}} U_2 \mathbf{m}}{\gamma_u \omega U_{21}} \arcsin\left[\frac{U_{21}}{\sqrt{1+U_2^2}} \sin\left(\frac{\omega t'}{\gamma_u} - \underline{\varphi}\right)\right] \tag{5.2}$$

$$\text{where} \quad \underline{\varphi} = \gamma_u \omega \mathbf{u}_0\cdot\mathbf{r}_0 - \gamma_{u_{\perp n}} \frac{U_1}{U_{12}} \mathbf{u}_0\cdot\mathbf{n} \arcsin\left[\frac{U_{12}}{\sqrt{1+U_1^2}} \cos\left(\frac{\omega t}{\gamma_u} - (\,)\right)\right] -$$
$$- \gamma_{u_{\perp m}} \frac{U_2}{U_{21}} \mathbf{u}_0\cdot\mathbf{m} \arcsin\left[\frac{U_{21}}{\sqrt{1+U_2^2}} \sin\left(\frac{\omega t}{\gamma_u} - (\,)\right)\right]$$

For vectors $\mathbf{n}$ and $\mathbf{m}$ from (5.2), the relation must be fulfilled $\mathbf{n}\cdot\mathbf{m}+\gamma_u^2(\mathbf{u}_0\cdot\mathbf{n})(\mathbf{u}_0\cdot\mathbf{m})=0$.

Velocity of a particle is

$$\mathbf{u}=\frac{\underline{\mathbf{U}}_f}{\gamma_u^2\left(1+\mathbf{u}_0\cdot\underline{\mathbf{U}}_f\right)}+\mathbf{u}_0 \qquad , \qquad u^2=1-\frac{1-\gamma_u^{-2}\underline{U}_f^2-\left(\mathbf{u}_0\cdot\underline{\mathbf{U}}_f\right)^2}{\gamma_u^2\left(1+\mathbf{u}_0\cdot\underline{\mathbf{U}}_f\right)^2}<1$$

$$\text{where}\quad \underline{\mathbf{U}}_f=\frac{\gamma_{u_{\perp n}}U_1\sin\left(\gamma_u^{-1}\omega t-\underline{\varphi}\right)\mathbf{n}-\gamma_{u_{\perp m}}U_2\cos\left(\gamma_u^{-1}\omega t-\underline{\varphi}\right)\mathbf{m}}{\sqrt{1+\left[U_1\sin\left(\gamma_u^{-1}\omega t-\underline{\varphi}\right)\right]^2+\left[U_2\cos\left(\gamma_u^{-1}\omega t-\underline{\varphi}\right)\right]^2}}$$

Equation (5.2) includes time with a decelerating coefficient of $1/\gamma_u$. The fractal phase $\underline{\varphi}$ consists of a constant and a periodic parts. The periodic part does not exceed the modulus of $\arcsin(0.5)$ and does not affect the magnitude of the main oscillation period, but it leads to shape distortion and the appearance of an infinite number of higher harmonics.

The relativistic equation of rotation of a particle (5.2) can be transformed to the form with mutually perpendicular vectors $\mathbf{n}$ and $\mathbf{n}_\perp$

$$\mathbf{r}=\mathbf{r}_0+\mathbf{u}_0t-\frac{\gamma_{u_{\perp n}}U_1\mathbf{n}}{\gamma_u\omega U_{12}}\arcsin\left[\frac{U_{12}}{\sqrt{1+U_1^2}}\cos\left(\frac{\omega t}{\gamma_u}-\underline{\varphi}\right)\right]-$$
$$-\frac{\gamma_{u_{\perp m}}U_2}{\gamma_u\omega U_{21}}\left\{(\mathbf{m}\cdot\mathbf{n})\mathbf{n}-[\mathbf{n}\times[\mathbf{n}\times\mathbf{m}]]\right\}\arcsin\left[\frac{U_{21}}{\sqrt{1+U_2^2}}\sin\left(\frac{\omega t'}{\gamma_u}-\underline{\varphi}\right)\right] \tag{5.3}$$

In the case of $(\mathbf{u}_0\cdot\mathbf{n})=u_0$ and $(\mathbf{u}_0\cdot\mathbf{n}_\perp)=0$ the law of motion (5.3) takes the form

$$\mathbf{r}=\mathbf{r}_0+\mathbf{u}_0t-\frac{U_1\mathbf{n}}{\gamma_u\omega U_{12}}\arcsin\left[\frac{U_{12}}{\sqrt{1+U_1^2}}\cos\left(\frac{\omega t}{\gamma_u}-\underline{\varphi}\right)\right]-$$
$$-\frac{U_2\mathbf{n}_\perp}{\omega U_{21}}\arcsin\left[\frac{U_{21}}{\sqrt{1+U_2^2}}\sin\left(\frac{\omega t}{\gamma_u}-\underline{\varphi}\right)\right] \tag{5.4}$$

$$\text{where}\quad \underline{\varphi}=\gamma_u\omega\mathbf{u}_0\cdot\mathbf{r}_0-\frac{u_0U_1}{U_{12}}\arcsin\left[\frac{U_{12}}{\sqrt{1+U_1^2}}\cos\left(\frac{\omega t}{\gamma_u}-(\,)\right)\right]$$

The amplitude of the particle's oscillations along the direction of velocity $\mathbf{u}_0$ decreases by a factor of $1/\gamma_u$, but does not change in the perpendicular direction.

In the case of circular rotation, the transition equation of motion (5.1) takes the form

$$\mathbf{r}' = \mathbf{r}_0 - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{r}_0) - \mathbf{v}t' - \frac{U_w}{\omega\sqrt{1+U_w^2}}\left[\mathbf{n} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{n})\right]\cos\left(\frac{\omega t'}{\gamma_v} + \underline{\varphi}'\right) - \\ - \frac{U_w}{\omega\sqrt{1+U_w^2}}\left[\mathbf{m} - \frac{\gamma_v \mathbf{v}}{\gamma_v + 1}(\mathbf{v}\cdot\mathbf{m})\right]\sin\left(\frac{\omega t'}{\gamma_v} + \underline{\varphi}'\right)$$

$$\text{where} \quad \underline{\varphi}' = \underline{\omega\mathbf{v}\cdot\mathbf{r}_0 - \frac{U_w \mathbf{v}\cdot\mathbf{n}}{\sqrt{1+U_w^2}}\cos\left(\frac{\omega t'}{\gamma_v} + (\,)\right) - \frac{U_w \mathbf{v}\cdot\mathbf{m}}{\sqrt{1+U_w^2}}\sin\left(\frac{\omega t'}{\gamma_v} + (\,)\right)}$$

The standard form of the law of relativistic circular rotation and uniform displacement is

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0 t - \frac{\gamma_{u_{\perp n}} U_w \mathbf{n}}{\gamma_u \omega\sqrt{1+U_w^2}}\cos\left(\frac{\omega t}{\gamma_u} - \underline{\varphi}\right) - \frac{\gamma_{u_{\perp m}} U_w \mathbf{m}}{\gamma_u \omega\sqrt{1+U_w^2}}\sin\left(\frac{\omega t}{\gamma_u} - \underline{\varphi}\right)$$

$$\text{where} \quad \underline{\varphi} = \underline{\gamma_u \mathbf{u}_0\cdot\mathbf{r}_0 - \frac{\gamma_{u_{\perp n}} U_w}{\sqrt{1+U_w^2}}\mathbf{u}_0\cdot\mathbf{n}\cos\left(\frac{\omega t}{\gamma_u} - (\,)\right) - \frac{\gamma_{u_{\perp m}} U_w}{\sqrt{1+U_w^2}}\mathbf{u}_0\cdot\mathbf{m}\sin\left(\frac{\omega t}{\gamma_u} - (\,)\right)}$$

6 Transformation of momentum, energy and force when transitioning to a new reference frame

Let the law of motion of a particle in the frame S have the form

$$\mathbf{R} = \mathbf{r}_0 + \int \mathbf{u}dt \quad , \quad \mathbf{u} = \frac{\mathbf{u}_w + \mathbf{U}_b}{\sqrt{1 + (\mathbf{u}_w + \mathbf{U}_b)^2}} \quad , \qquad \mathbf{U}_b = const \quad , \quad \mathbf{u}_w = \mathbf{u}_w(t)$$

The momentum $\mathbf{p}$ and energy E in the frame S are equal to

$$\mathbf{p}(t) = M(\mathbf{u}_w + \mathbf{U}_b) = \mathbf{p}_w + \mathbf{p}_b \quad , \quad E(t) = M\sqrt{1 + (\mathbf{u}_w + \mathbf{U}_b)^2}$$

Well-known equations for the transformation of momentum, energy, and force

$$\mathbf{p}' = \mathbf{p} + \Gamma_v \mathbf{v}(\mathbf{v}\cdot\mathbf{p}) - \gamma_v E\mathbf{v} \quad , \quad E' = \gamma_v(E - \mathbf{v}\cdot\mathbf{p}) \tag{6.1}$$

$$\mathbf{F}' = \frac{\mathbf{F} - \gamma_v\,(\mathbf{F}\cdot\mathbf{u})\mathbf{v} + \Gamma_v \mathbf{v}(\mathbf{v}\cdot\mathbf{F})}{\gamma_v(1 - \mathbf{v}\cdot\mathbf{u})} \tag{6.2}$$

and transformation formulas for electric and magnetic fields

$$\mathbf{E}' = \gamma_v \mathbf{E} - \Gamma_v \mathbf{v}(\mathbf{v}\cdot\mathbf{E}) + \gamma_v[\mathbf{v}\times\mathbf{B}] \quad , \quad \mathbf{B} = \gamma_v \mathbf{B} - \Gamma_v \mathbf{v}(\mathbf{v}\cdot\mathbf{B}) - \gamma_v[\mathbf{v}\times\mathbf{E}] \tag{6.3}$$

The straightforward application of these transformations is valid only if they do not involve time—that is, if the particles in frame S move at constant velocities and the forces of their interaction are constant over time. If the particle velocities change over time and their interaction is not stationary, then as a result of the transformations (6.1) - (6.3) for the frame S', quantities are obtained that are functions of time t from the frame S. To express these as functions of the time t', the equations (6.1)–(6.3) must be supplemented by the direct Lorentz transformation for time: $t' = \gamma_v[t - \mathbf{v}\cdot\mathbf{R}(t)]$. From the direct transformation of time we find the inverse function $t' = t'(t)$, which is almost always a fractal

$$t = \underline{t'/\gamma_v + \mathbf{v}\cdot\mathbf{R}(\,)} = t'/\gamma_v + \underline{\varphi'} \qquad , \qquad \underline{\varphi'} = \underline{\mathbf{v}\cdot\mathbf{R}[t'/\gamma_v + (\,)]} \tag{6.4}$$

Substituting (6.4) into equations (6.1) – (6.3), we obtain the required transformations for momentum, energy, and force.

$$\underline{\mathbf{p}}'(t') = \underline{\mathbf{p}} + \Gamma_v\mathbf{v}(\mathbf{v}\cdot\underline{\mathbf{p}}) - \gamma_v\underline{E}\mathbf{v} \quad , \quad \underline{E}'(t') = \gamma_v\big(\underline{E} - \mathbf{v}\cdot\underline{\mathbf{p}}\big) \tag{6.5}$$

$$\underline{\mathbf{F}}'(t') = \frac{\underline{\mathbf{F}} - \gamma_v\,(\underline{\mathbf{F}}\cdot\underline{\mathbf{u}})\,\mathbf{v} + \Gamma_v\mathbf{v}(\mathbf{v}\cdot\underline{\mathbf{F}})}{\gamma_v(1 - \mathbf{v}\cdot\underline{\mathbf{u}})} = \gamma_v\underline{\mathbf{F}} - \Gamma_v\mathbf{v}(\mathbf{v}\cdot\underline{\mathbf{F}}) - \gamma_v\,[\underline{\mathbf{u}}'\times[\mathbf{v}\times\underline{\mathbf{F}}]] \tag{6.6}$$

$$\underline{\mathbf{p}} = \mathbf{p}_w\left(t'/\gamma_v + \underline{\varphi'}\right) + \mathbf{p}_b \quad , \quad \underline{E} = M\sqrt{1 + \left[\mathbf{u}_w\left(t'/\gamma_v + \underline{\varphi'}\right) + \mathbf{U}_b\right]^2} \quad , \quad \underline{\mathbf{F}} = \mathbf{F}\left(t'/\gamma_v + \underline{\varphi'}\right)$$

and the transformation equations for the electric and magnetic field vectors

$$\underline{\mathbf{E}}'(t') = \gamma_v\underline{\mathbf{E}} - \Gamma_v\mathbf{v}(\mathbf{v}\cdot\underline{\mathbf{E}}) + \gamma_v[\mathbf{v}\times\underline{\mathbf{B}}] \quad , \quad \underline{\mathbf{B}}'(t') = \gamma_v\underline{\mathbf{B}} - \Gamma_v\mathbf{v}(\mathbf{v}\cdot\underline{\mathbf{B}}) - \gamma_v[\mathbf{v}\times\underline{\mathbf{E}}] \tag{6.7}$$

$$\underline{\mathbf{E}} = \mathbf{E}\left(t'/\gamma_v + \underline{\varphi'}\right) \qquad , \qquad \underline{\mathbf{B}} = \mathbf{B}\left(t'/\gamma_v + \underline{\varphi'}\right)$$

When moving from the frame S to the frame S' , the vectors change their direction and modulus. If we write equations (6.5) - (6.7) taking into account the change in vectors in the frame S' , then they take on a simpler form. Let us consider unit vectors $\mathbf{n}'$ and $\mathbf{m}'$ in frame S'.

$$\mathbf{n}' = \frac{\gamma_{v_{\perp n}}}{\gamma_v}\left[\mathbf{n} + \Gamma_v(\mathbf{v}\cdot\mathbf{n})\mathbf{v}\right] \quad , \qquad \gamma_{v_{\perp n}} = \frac{1}{\sqrt{1 - (v)^2 + (\mathbf{v}\cdot\mathbf{n})^2}} \quad , \quad \mathbf{n} = \frac{\mathbf{u}_w}{u_w}$$

$$\mathbf{m}' = \frac{\gamma_{v_{\perp m}}}{\gamma_v}\left[\mathbf{m} + \Gamma_v(\mathbf{v}\cdot\mathbf{m})\mathbf{v}\right] \quad , \qquad \gamma_{v_{\perp m}} = \frac{1}{\sqrt{1 - (v)^2 + (\mathbf{v}\cdot\mathbf{m})^2}} \quad , \quad \mathbf{m} = \frac{\mathbf{U}_b}{U_b}$$

Then equations (6.5) can be rewritten as

$$\underline{\mathbf{p}}'(t') = \gamma_{v_{n'}}\underline{p}_w\mathbf{n}' + \gamma_{v_{m'}}\underline{p}_b\mathbf{m}' - \gamma_v\mathbf{v}\underline{E} \quad , \quad \underline{E}'(t') = \gamma_v\underline{E} - \gamma_{v_{n'}}\underline{p}_w\mathbf{v}\cdot\mathbf{n}' - \gamma_{v_{m'}}\underline{p}_b\mathbf{v}\cdot\mathbf{m}'$$

$$\gamma_{v_{n'}} = \frac{1}{\sqrt{1-(\mathbf{v}\cdot\mathbf{n}')^2}} \quad , \quad \gamma_{v_{m'}} = \frac{1}{\sqrt{1-(\mathbf{v}\cdot\mathbf{m}')^2}}$$

Let us take unit vectors $\mathbf{n}^*$ and $\mathbf{m}^*$ in the frame S'

$$\mathbf{n}^* = \gamma_{v_n}\left[\mathbf{n} - \frac{\gamma_v\mathbf{v}}{\gamma_v+1}(\mathbf{v}\cdot\mathbf{n})\right] \quad , \quad \mathbf{m}^* = \gamma_{v_m}\left[\mathbf{m} - \frac{\gamma_v\mathbf{v}}{\gamma_v+1}(\mathbf{v}\cdot\mathbf{m})\right]$$

Then equations (6.6) – (6.7) take the form

$$\underline{\mathbf{F}}'(t') = \gamma_{v_{\perp n^*}}\underline{F}\left\{\mathbf{n}^* - [\underline{\mathbf{u}}'\times[\mathbf{v}\times\mathbf{n}^*]]\right\} \quad , \quad \underline{\mathbf{u}}'(t') = \frac{\underline{\mathbf{p}}'}{\underline{E}'}$$

$$\underline{\mathbf{E}}'(t') = \gamma_{v_{\perp n^*}}\underline{E}\mathbf{n}^* + \gamma_{v_{\perp m^*}}\underline{B}\,[\mathbf{v}\times\mathbf{m}^*] \quad , \quad \underline{\mathbf{B}}'(t') = \gamma_{v_{\perp m^*}}\underline{B}\mathbf{m}^* + \gamma_{v_{\perp n^*}}\underline{E}\,[\mathbf{v}\times\mathbf{n}^*]$$

7 Transformation of forces in uniform motion

Let particles 1 and 2 move uniformly in the reference frame S $\mathbf{R}_1 = \mathbf{r}_1 + \mathbf{u}_1 t$, $\mathbf{R}_2 = \mathbf{r}_2 + \mathbf{u}_2 t$. Distance between particles $\mathbf{R}_{21} = \mathbf{R}_1 - \mathbf{R}_2$. The equation of motion of particle 1 in the reference frame S'

$$\mathbf{R}_1' = \mathbf{r}_1 + \frac{\gamma_v\mathbf{u}_1 - \Gamma_v\mathbf{v}}{\gamma_v(1-\mathbf{v}\cdot\mathbf{u}_1)}(\mathbf{v}\cdot\mathbf{r}_1) + \mathbf{u}_1' t' \quad , \quad \mathbf{u}_1' = \frac{\mathbf{u}_1 - \gamma_v\mathbf{v} + \Gamma_v\mathbf{v}(\mathbf{v}\cdot\mathbf{u}_1)}{\gamma_v(1-\mathbf{v}\cdot\mathbf{u}_1)} \tag{7.1}$$

The equation of motion of particle 2 in the reference frame S' is given by a similar expression.

The force of action of particle 2 on particle 1 in the frame S is equal to

$$\mathbf{F}_{21} = \frac{\alpha\,(1-u_2^2)}{\left[R_{21}^2\,(1-u_2^2) + (\mathbf{R}_{21}\cdot\mathbf{u}_2)^2\right]^{3/2}}\left\{\mathbf{R}_{21} + [\mathbf{u}_1\times[\mathbf{u}_2\times\mathbf{R}_{21}]]\right\} \quad , \quad \text{where} \quad \alpha - \begin{matrix}\text{the charge}\\ \text{coefficient}\end{matrix} \tag{7.2}$$

At time t in frame S particle 2 acts on particle 1, and particle 1 acts on particle 2. In frame S' the first event occurs at time $t_1' = \gamma_v\,[t - \mathbf{v}\cdot\mathbf{R}_1(t)]$, and the second at time $t_2' = \gamma_v\,[t - \mathbf{v}\cdot\mathbf{R}_2(t)]$. In this case, the event of the action of particle 2 on particle 1 in the frame S' at the moment t_1' is Lorentz-invariant for particle 1, but not Lorentz-invariant for particle 2. The distance from particle

2 to particle 1 in frame S' at time t'_1 is equal to

$$\mathbf{R}'_{21} = \mathbf{R}'_1(t'_1) - \mathbf{R}'_2(t'_1) = \mathbf{R}_{21} + \frac{\gamma_v \mathbf{u}_2 - \Gamma_v \mathbf{v}}{\gamma_v(1 - \mathbf{v}\cdot\mathbf{u}_2)}(\mathbf{v}\cdot\mathbf{R}_{21})$$

Similarly, the event of the action of particle 1 on particle 2 occurs at time t'_2 , when the distance between particles is $\mathbf{R}'_{12}(t'_2)$.

Let us find the law of action of particle 2 on particle 1 in the reference frame S' . We will express the quantities from the reference frame S in terms of the quantities from the reference frame S' .

$$\mathbf{R}_{21} = \mathbf{R}'_{21} - \frac{\gamma_v \mathbf{u}'_2 + \Gamma_v \mathbf{v}}{\gamma_v(1 + \mathbf{v}\cdot\mathbf{u}'_2)}(\mathbf{v}\cdot\mathbf{R}'_{21}) \quad , \quad \mathbf{u}_2 = \frac{\mathbf{u}'_2 + \gamma_v \mathbf{v} + \Gamma_v \mathbf{v}(\mathbf{v}\cdot\mathbf{u}'_2)}{\gamma_v(1 + \mathbf{v}\cdot\mathbf{u}'_2)} \quad ,$$

$$R^2_{21} = (R'_{21})^2 - \frac{2\mathbf{v}\cdot\mathbf{R}'_{21}}{1 + \mathbf{v}\cdot\mathbf{u}'_2}(\mathbf{u}'_2\cdot\mathbf{R}'_{21}) - \frac{1 - (u'_2)^2}{(1 + \mathbf{v}\cdot\mathbf{u}'_2)^2}(\mathbf{v}\cdot\mathbf{R}'_{21})^2 \quad , \quad \mathbf{v}\cdot\mathbf{R}_{21} = \frac{\mathbf{v}\cdot\mathbf{R}'_{21}}{\gamma_v(1 + \mathbf{v}\cdot\mathbf{u}'_2)}$$

and we substitute them into transformations for the force $\mathbf{F}'_{21}$ (7.3)

$$\mathbf{F}'_{21} = \frac{\mathbf{F}_{21} - \gamma_v \mathbf{v}(\mathbf{u}_1\cdot\mathbf{F}_{21}) + \Gamma_v \mathbf{v}(\mathbf{v}\cdot\mathbf{F}_{21})}{\gamma_v(1 - \mathbf{v}\cdot\mathbf{u}_1)} \tag{7.3}$$

As a result, we obtain equation (7.2) expressed in terms of primed quantities — a result supported by both theoretical electrodynamics and experimental data. The transformation of forces is calculated for an arbitrary instant in time t. Consequently, by knowing the interaction forces at any given time t in a reference frame S, we can determine the corresponding forces in a reference frame S'. This calculation demonstrates that the force exerted by charge q_1 on charge q_2 and the force exerted by charge q_2 on charge q_1—which are simultaneous in one reference frame S—become non-simultaneous in a second reference frame S'.

8 Calculation of retarded distances

Let the equation of motion of a particle be given by the function $\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{R}_f(t) = \mathbf{R}_0 + R_f(t)\mathbf{n}$. Then $\mathbf{r} -$ is the distance from the particle to point A at time t; $\mathbf{r}_p -$ is the retarded distance from the particle's past location at time (t–r_p) to point A.

The modulus r_p is calculated using a fractal function

$$\underline{r_p} = \underline{\sqrt{r^2 + 2\mathbf{r}\cdot\mathbf{R}(t) + R^2(t) - 2\mathbf{r}\cdot\mathbf{R}(t - (\,)) - 2\mathbf{R}(t)\cdot\mathbf{R}(t - (\,)) + R^2(t - (\,))}} \tag{8.1}$$

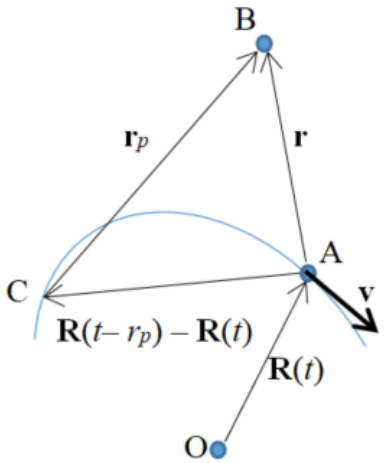


Figure 1: Calculation of retarded distance

Under condition $r \gg |\mathbf{R}(t) - \mathbf{R}(t - r_p)|$, we obtain $\underline{r_p \cong r + \frac{\mathbf{r}}{r}\cdot[\mathbf{R}(t) - \mathbf{R}(t - (\,))]}$

If the function of motion $\mathbf{R}(t)$, at the time t of finding the retarded distance $\mathbf{r}_p$, corresponds to the condition $\mathbf{R}(t) = 0$, then expression (8.1) is significantly simplified

$$\mathbf{r}_p = \mathbf{r} - \mathbf{R}(t - r_p) \quad , \quad \underline{r_p = \sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{R}(t - (\,)) + R^2(t - (\,))}} \tag{8.2}$$

Thus, functions with retarded time can be transformed to a fractal form and their values can be found using computer technology.

The retarded distance can be calculated for the case of uniform motion using the formula (8.2).

$$\mathbf{r}_p = \mathbf{r} + r_p\mathbf{u}_0 \quad , \quad \underline{r_p = \sqrt{r^2 + 2\mathbf{u}_0\cdot\mathbf{r}\cdot(\,) + u_0^2\cdot(\,)^2}}$$

The fractal function sums up to the function (8.3), which is the solution to the quadratic equation for r_p .

$$r_p = \frac{1}{1 - u_0^2}\left[\mathbf{u}_0\cdot\mathbf{r} + \sqrt{r^2(1 - u_0^2) + (\mathbf{u}_0\cdot\mathbf{r})^2}\right] \tag{8.3}$$

9 Transformation of forces in relativistic uniformly accelerated motion

In the case of accelerated motion, electromagnetic forces depend on retarded distances, as well as on velocity and acceleration at retarded time points. The method of calculating retarded distances was discussed in the previous section.

In the reference frame S, let particle 1 move at a constant velocity $\mathbf{u}_1$, and particle 2 move relativistic

uniformly accelerated. The equation of motion of particle 1 in the reference frame S $\mathbf{R}_1 = \mathbf{r}_1 + \mathbf{u}_1 t$. The equation of motion of particle 2 in the reference frame S

$$\mathbf{R}_2 = \mathbf{r}_2 + \frac{\mathbf{n}}{W}\left(\sqrt{1+W^2t^2}-1\right)$$

The force of the action of particle 1 on particle 2 in the reference frame S is found by the formula (7.2). The force of the action of particle 2 on particle 1 in the reference frame S is obtained by applying the formula from electrodynamics for the electromagnetic field of an arbitrarily moving charge (Lienard-Wiechert solutions of Maxwell's equations). The retarded distances are calculated using the formula (8.2).

The law of motion of particle 1 in the reference frame S' is given by equation (7.1) . The law of motion of particle 2 in the reference frame S' is

$$\mathbf{R}'_2 = \mathbf{r}_2 - \frac{\gamma_v \mathbf{v}}{\gamma_v+1}(\mathbf{v}\cdot\mathbf{r}_2) + \gamma_{v_n}(\mathbf{v}\cdot\mathbf{r}_2)\,(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}' - \left[\mathbf{v} - \frac{\gamma_{v_n}}{\gamma_v}(\mathbf{v}\cdot\mathbf{n})\mathbf{n}'\right]t' + \\ + \frac{\mathbf{n}'}{W}\left[\sqrt{1+\gamma_{v_n}^2(\gamma_v^{-1}Wt' + W\mathbf{v}\cdot\mathbf{r}_2 - \mathbf{v}\cdot\mathbf{n})^2} - \gamma_{v_n}\right]$$

The force of the action of particle 2 on particle 1 in the frame S corresponds to the force in the frame S' at the moment of time $t'_1 = \gamma_v\left[t - \mathbf{v}\cdot\mathbf{R}_1(t)\right]$. The distance between particles is $R'_{21} = R'_1(t'_1) - R_2(t'_1)$. For a given distance, the force of action of particle 2 on particle 1 in the frame S' is found using formulas from electrodynamics (Lienard-Wiechert solution of Maxwell's equations). The value of the force $\mathbf{F}'_{21}(t)$ coincides with the value of the force $\mathbf{F}'_{21}(t)$ calculated from the Lorentz transformations for forces (7.3). A similar calculation of the force $\mathbf{F}'_{12}(t)$ in the frame S' at the moment of time $t'_2 = \gamma_v\left[t - \mathbf{v}\cdot\mathbf{R}_2(t)\right]$ coincides with the value calculated using the Lorentz force transformation formula.

10 Consequences of Lorentz transformations

10.1 Dynamic compression of bodies

Let two particles begin to move at an accelerated pace along the X-axis from a state of rest. After a certain period of time, their velocities reach a threshold value of u_0, at which they stabilize. Due to the synchronicity of the movement of the particles and due to the lack of their interaction, the distances between them remain unchanged all the time. That is, there is no effect of reducing the

distance between particles when they move simultaneously at relativistic speeds. Next, we write down the law of relativistic rotation and displacement with constant velocity when the velocity $\mathbf{u}_0$ is parallel to the unit vector $\mathbf{n}_1$ (5.4) . It follows from the law of motion that the trajectory of rotation is compressed by a factor of $1/\gamma_u$ along the line of constant velocity $\mathbf{u}_0$. This is due to the fact that the source of force causing rotation in the initially-own reference frame in the current reference frame moves at a constant velocity $\mathbf{u}_0$ and, accordingly, interacts differently with a moving charge. Namely, in such a way that the trajectory of its rotation is compressed. A similar conclusion can be drawn from the general equation of motion of the particle (3.12) - (3.13) .

Thus, it can be assumed that the solid body will also contract along the line of motion with a constant speed by γ_u – times. That is, the compression of solid bodies is a dynamic effect determined by a change in the interaction of charges. At one time, Lorentz [2] made a similar conclusion based on the transformation of the spherical equipotential surface of the electrostatic field of a stationary electron into an ellipsoid surface of a moving electron.

10.2 Dynamic dilation of proper time of moving bodies

The proper time of a moving particle characterizes the rate of internal processes. The proper time of a moving particle is shown by the clock associated with it. Any periodic motion accompanying the main motion of a particle can be used as a clock. For example, the equation of motion (5.2) can be considered as a uniform motion of a particle and a description of the course of the clock associated with it. If we write down the equation of motion of a particle consisting of an arbitrary motion of the center of mass $\mathbf{r}_f$ and periodic oscillations relative to the center of mass, then the oscillations will characterize the flow of proper time during particle motion according to the $\mathbf{r}_f$ law. The slowing down of a moving clock compared to a coordinate clock is a consequence of a change in the nature of the internal interactions of a moving particle. This is the same dynamic effect as shortening the length of a body moving at a constant speed. A. Einstein once gave a similar explanation for the slowing down of proper time in moving particles [3]. At the same time, he relied on experimental data on a decrease in the frequency of radiation associated with internal processes in moving particles. This paper mathematically proves the existence of such a slowdown (3.11) .

10.3 Metrological change in distance

Let us consider the change in the distance between synchronously uniformly moving particles from the reference frame S during the transition to the reference frame S' moving along the X- axis with a

constant velocity $\mathbf{v}$. Let there be two independent (non-interacting) particles in the reference frame S, the particles move along the X-axis with a constant velocity $u:\ X_1 = x_1 + ut\ ,\ X_2 = x_2 + ut$. In the reference frame S', the laws of their motion have the form [4]

$$X'_1 = \frac{x_1}{\gamma_v(1-vu)} + \frac{u-v}{1-vu}t'\ \ ,\ \ X'_2 = \frac{x_2}{\gamma_v(1-vu)} + \frac{u-v}{1-vu}t'\ \ ,\ \ X'_{12} = \frac{x_{12}}{\gamma_v(1-vu)} \tag{10.1}$$

It can be concluded that the length of the distance between particles 1 and 2 is greatest when they are stationary in the reference frame, and the moving distance always has a shorter length. When switching from reference frame S to reference frame S', depending on how the speed of the distance changes, the distance itself can either increase or decrease.

If a solid body moves with a velocity $\mathbf{u}$ in a reference frame S, the change in its dimensions upon transitioning to a frame S' is described by the same formula (10.1). However, as follows from (5.4), the body undergoes Lorentz contraction in frame S. This happened because the elements of the structure of a solid body moving at the speed $\mathbf{u}$ begin to interact with each other differently. As a result, the body is compressed along the direction of the velocity $\mathbf{u}$ by a factor of γ_u. In this case, it is a purely dynamic effect. When moving from the frame S to the frame S' , as a result of the transformation of the interaction forces, the elements of the structure of the solid interact with each other in a new way. On the other hand, the principle of the equivalence of all IRFs implies that the physical state of a body must remain the same across them; no stresses or deformations should arise merely from a change in the reference frame. This apparent contradiction is resolved by the fact that, although the standards for length and time in frames S and S' are defined according to identical rules, they are not equal to one another due to the relative motion between the frames. Therefore, the change in the nature of the interaction of the structural elements of a solid body and its dimensions upon transition to the frame S' is a purely metrological, and not a dynamic effect. Any change in distances, time, and forces when switching to another frame of reference is not associated with any physical processes – it is a purely metrological effect, the effect of switching to other standards of length and time.

Formula (10.1), which describes the change in the magnitude of the distance between particles 1 and 2 when transitioning from frame S to frame S', relates two distinct distances, as follows from the Lorentz transformations. According to the Lorentz transformations, a distance in frame S corresponds to a distance in frame S' whose endpoints, in the general case, do not exist at the same instant in time. However, only distances whose endpoints exist at the same common instant in time possess physical meaning. Therefore, a formula describing the change in the magnitude of the distance must

incorporate the distance r_{12} at a given time t in frame S and the distance r'_{12} at a corresponding synchronous time $t' = t/\gamma_v$ in frame S'. By performing transformations to move from a distance at different times to a distance at the same time, we obtain the standard formula for transforming the modulus of distances (10.1).

10.4 The general case of metrological distance change

Consider the case of the distance between synchronously accelerating particles. Let there be two independent (non-interacting) particles in the reference frame S which begin to move synchronously according to the law

$$X_1 = x_1 + \frac{1}{W}\left(\sqrt{1+W^2t^2}-1\right) \quad , \quad X_2 = x_2 + \frac{1}{W}\left(\sqrt{1+W^2t^2}-1\right)$$

Since the movement is synchronous, the distance between the particles in the reference frame S remains constant at any given time and at any speed. Let's make the transition to a reference frame S' moving along the X- axis with a velocity of $\mathbf{v}$. The law of motion of particle 1 in frame S' has the form

$$X'_1 = \gamma_v x_1 + \frac{1}{W}\left\{\sqrt{1+(Wt'+\gamma Wvx_1-\gamma v)^2}-\gamma_v\right\} \qquad , \qquad U'_1 = \frac{Wt'+\gamma Wvx_1-\gamma v}{\sqrt{1+(Wt'+\gamma Wvx_1-\gamma v)^2}}$$

The law of motion for particle 2 has a similar form. The distance between the particles in frame S' is then given by the equation

$$X'_{12} = \gamma_v x_{12} + \frac{1}{W}\left(\sqrt{1+(Wt'+\gamma_v Wvx_2-\gamma_v v)^2}-\sqrt{1+(Wt'+\gamma_v Wvx_1-\gamma_v v)^2}\right) \quad , \qquad (10.2)$$

$$\lim_{W\to 0} X'_{12} = \frac{x_{12}}{\gamma_v} \qquad , \qquad \lim_{t\to+\infty} X'_{12} = \frac{x_{12}}{\gamma_v(1-v)} \qquad , \qquad \lim_{t\to-\infty} X'_{12} = \frac{x_{12}}{\gamma_v(1+v)}$$

The plot of the dependence $X'_{12}(t')$ is constructed for the following values:
$W = 0,1$; $v = 0,4$ $x_1 = 1$; $x_2 = 4$.
This case is interesting for two reasons. First, the velocities u'_1 and u'_2 vary over time, with $u'_2 > u'_1$. In frame S, particles 1 and 2 are subject to identical forces, such that $u_2 = u_1$; however, in frame S', the force acting on particle 2 is $F'_2 > F'_1$. Thus, when transforming forces, in the general case, the initial location of the particle ($x_2 > x_1$) is also important, which is not clearly evident from the force transformation equation. Second, as $t \to +\infty$, the velocities $u_2 = u_1 \to c$ (where $c = 1$), while as $t \to -\infty$, the velocities $u_2 = u_1 \to -c$. Then the value of X'_{12} (10.2) coincides with the value from

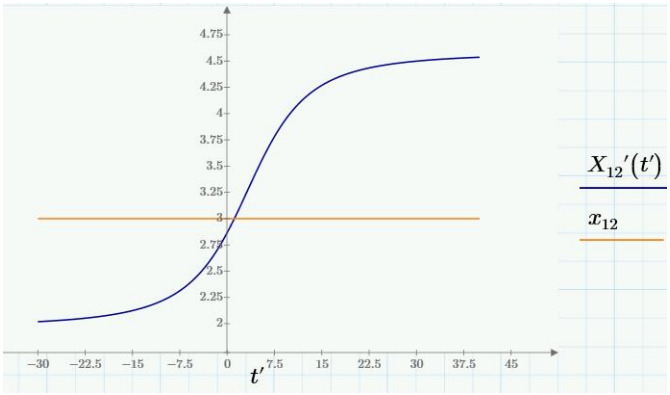


Figure 2: Change in distance upon transition to the frame S'

the formula (10.1) .

11 Discussion

For the first time, the article derives an analytical expression for the relativistic motion of a particle in an IRF (3.10) and (3.13). Until now, the law of relativistic motion of a particle has been obtained only by numerical methods, in the form of graphs. Special cases of relativistic motion of a particle are considered: uniformly accelerated motion with uniform displacement and periodic motion with uniform displacement. Relativistic expressions for the equations of motion, momentum, energy, and force generally have a fractal form. The ability to work with analytical expressions for the laws of particle motion at relativistic speeds can be compared to the transition from Euclidean geometry to analytical geometry. In Euclidean geometry, geometric constructions play the primary role, whereas in analytical geometry, their role is purely auxiliary.

The article applies a method for calculating retarded distances (electrodynamics) using fractal functions.

Infinite series are widely used in mathematics and physics. Fractals, as a certain analogue of infinite series, also have great potential for their application in many areas of physics and mathematics. The use of fractals within the framework of special relativity has demonstrated the promise and viability of this approach. The metrological understanding of Lorentz transformations makes it possible to begin developing a theory of non inertial reference frames. A. Einstein's general theory of relativity is a theory of gravity.

Bibliographic references

References

[1] *S. S. Stepanov, Relativistic world, Berlin, Boston: De Gruyter, 2018*

[2] *H. A. Lorentz, The theory of electrons and its applications to the phenomena of light and radiant heat. A course of lectures delivered in Columbia University, New York, in March and April 1906.*

[3] *A. Einstein, Principe de relativite et ses consequences dans la physique moderne. Arch. sci. phys. Natur., ser. 4, 1910.*

[4] *S. D. Agashe, Is the Lorentz contraction inevitable in the special theory of relativity ?, https://arxiv.org/abs/0904.1913*